

# Some New Constructions of Minimal Neighbour Designs in Circular Blocks

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Neighbour balanced designs are useful for cases where the performance of a treatment is affected by the treatments applied to its neighbouring plots. Among these designs, minimal Neighbour balanced designs are preferred because these are economical. In this article, some generators are developed through the method of cyclic shifts to obtain minimal circular Neighbour balanced designs in blocks of (a) equal sizes, and (b) two different sizes.

**Key words:** *Neighbour design; Neighbour effects; Circular blocks; Minimal designs.*

**Mathematics Subject Classification (2010):** 05B05; 62K10; 62K05.

## 1. Introduction

In the field of experimental design, the main problem is to control extraneous effects. The risk of experimental error increases if the sources of variation cannot be controlled. Experiments are required to be designed so that the extraneous sources of variations can be controlled. Similarly, experimental designs are not preferred in situations where the performance of a treatment is affected by the treatments applied to its neighbouring plots. Neighbour designs were introduced by Rees (1967) in minimal circular blocks where any two distinct treatments appear as neighbours exactly once. Das and Saha (1976) presented different methods to construct minimal neighbour designs. A design in which each pair of non-identical treatments appears once in adjacent plots of the same block is called minimal neighbour design. If every treatment occurs once as the neighbour of every other treatment, either on the right or on the left, the design is called minimal non-directionally neighbour balanced (Azais *et al.* 1993). Azais *et al.* (1993) stated that the linear block can be used as the circular block, using a border plot with the condition that the treatments of border plots must be the same as the treatments of the inner plot at the other end of the block. Iqbal *et al.* (2009) proposed several neighbour designs using cyclic shifts. Akhtar *et al.* (2010), Ahmed and Akhtar (2011a), Ahmed and Akhtar (2011b) and Shehzad *et al.* (2011) constructed neighbour designs for some cases. Aldred *et al.* (2014) defined three different types of circular neighbour balanced designs (CNBDs) which are balanced at distance one and two. They considered the arrangement of

treatments on a circle or single line either in space or time. Jaggi *et al.* (2017) provided several methods to construct circular neighbour balanced and circular partially neighbour balanced block designs. In this paper, some new generators are developed to obtain minimal CNBDs in blocks of (a) equal sizes, and (b) two different sizes.

The article is organized as follows. In Section 2, method of cyclic shifts (Rule I) is explained for the construction of minimal CNBDs. In Section 3, some generators are developed to obtain minimal CNBDs in blocks of equal sizes. In Section 4, some generators are developed to obtain minimal CNBDs in blocks of two different sizes.

## 2. Method of Cyclic Shifts

Method of cyclic shifts (Rule I) is explained here for the construction of minimal CNBDs. This method was introduced by Iqbal (1991).

**Rule I:** Let  $S_j = [q_{i1}, q_{i2}, \dots, q_{i(k-1)}]$  be the sets of shifts, where  $1 \leq q_{ij} \leq v-1$ . If each of  $1, 2, \dots, v-1$  appears once in  $S^*$ , where  $S^* = [q_{i1}, q_{i2}, \dots, q_{i(k-1)}, (q_{i1}+q_{i2}+\dots+q_{i(k-1)}) \bmod v, v-q_{i1}, v-q_{i2}, \dots, v-q_{i(k-1)}, v-[(q_{i1}+q_{i2}+\dots+q_{i(k-1)}) \bmod v]]$  then  $S_j$  will provide minimal CNBD.

**Example 2.1:** Minimal CNBD is obtained from  $S = [1,3,4,9]$  for  $v = 11, k = 5$ .

**Proof:**  $S^* = [1,3,4,9,6,10,8,7,2,5]$ , Here each of  $1, 2, \dots, 10$  appears once. Hence  $S = [1,3,4,9]$  provide minimal CNBD for  $v = 11$  and  $k = 5$ . Design will be generated from these sets of shifts as follows.

Take  $v$  blocks for  $S$ . Allot  $0, 1, 2, \dots, v-1$  as the first unit element for each block respectively. Add the first element of  $S \pmod v$  to the each first unit element of first block to get second unit elements of second block. Similarly, add the second element of  $S \pmod v$  to the each second unit element to obtain the third unit elements of each block, and so on, see Table 1.

**Table 1:** Minimal CNBD generated from  $S = [1,3,4,9]$  for  $v = 11, k = 5$

Blocks										
B <sub>1</sub>	B <sub>2</sub>	B <sub>3</sub>	B <sub>4</sub>	B <sub>5</sub>	B <sub>6</sub>	B <sub>7</sub>	B <sub>8</sub>	B <sub>9</sub>	B <sub>10</sub>	B <sub>11</sub>
0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	1
4	5	6	7	8	9	10	1	2	3	4
8	9	10	1	2	3	4	5	6	7	8
6	7	8	9	10	1	2	3	4	5	6

### 3. Generators to Obtain Minimal CNBDs in Blocks of Equal Sizes

In this Section, some new generators are developed to obtain the  $i$  sets of shifts which provide the minimal CNBDs in blocks of equal sizes. These  $i$  sets of shifts are obtained as follows. In all these constructions  $m = (m-1)/2$ .

- Divide resultant elements of  $A$  given for required case into  $i$  groups of size  $k$  such that the sum of each group must be divisible by  $v$ .
- By deleting one element (any) from each group, we will get the required  $i$  sets of shifts.

**Generator 3.1:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+5$ ;  $l$  &  $u$ (integer)  $\geq 0$ .

Case I: If  $i = 8u+1$ , replace  $\frac{3iK+1}{8}$  with  $v - \frac{3iK+1}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case II: If  $i = 8u+3$ , replace  $\frac{7iK+7}{8}$  with  $v - \frac{7iK+7}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case III: If  $i = 8u+5$ , replace  $\frac{3iK+5}{8}$  with  $v - \frac{3iK+5}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case IV: If  $i = 8u+7$ ; replacing  $\frac{7iK+3}{8}$  with  $v - \frac{7iK+3}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.1.**  $S = [1, 3, 4, 9]$  provides minimal CNBD for  $v = 11$  and  $k = 5$ .

**Generator 3.2:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 4l+5$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 8u+2$ , replace  $\frac{5iK+6}{8}$  with  $v - \frac{5iK+6}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case II: If  $i = 8u+4$ , replace  $\frac{iK+4}{8}$  with  $v - \frac{iK+4}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case III: if  $i = 8u+6$ , replace  $\frac{5iK+2}{8}$  with  $v - \frac{5iK+2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case IV: If  $i = 8u+8$ , replace  $\frac{iK}{8}$  with  $v - \frac{iK}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.2.**  $S_1 = [1, 3, 5, 8]$  and  $S_2 = [2, 6, 11, 14]$  provide minimal CNBD for  $v = 21$  and  $k = 5$ .

**Generator 3.3:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+9$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 8u+1$ ; replacing  $\frac{3iK+5}{8}$  with  $v - \frac{3iK+5}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case II: If  $i = 8u+3$ ; replacing  $\frac{7iK+3}{8}$  with  $v - \frac{7iK+3}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case III: If  $i = 8u+5$ ; replacing  $\frac{3iK+1}{8}$  with  $v - \frac{3iK+1}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case IV: If  $i = 8u+7$ ; replacing  $\frac{7iK+7}{8}$  with  $v - \frac{7iK+7}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 3.3.**  $S = [1, 2, 3, 5, 6, 7, 8, 15]$  provides minimal CNBD for  $v = 19$  and  $k = 9$ .

**Generator 3.4:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+6$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 4u+1$ , replace  $\frac{5iK+2}{8}$  with  $v - \frac{5iK+2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case II: If  $i = 4u+3$ , replace  $\frac{5iK+6}{8}$  with  $v - \frac{5iK+6}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 3.4.**  $S = [1, 2, 5, 6, 9]$  provides minimal CNBD for  $v = 13$  and  $k = 6$ .

**Generator 3.5:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 4l+6$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 4u+2$ , replace  $\frac{iK+4}{8}$  with  $v - \frac{iK+4}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case II: If  $i = 4u+4$ , replace  $\frac{iK}{8}$  with  $v - \frac{iK}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.5.**  $S_1 = [1, 4, 5, 6, 23]$ ,  $S_2 = [3, 7, 8, 10, 13]$  provide minimal CNBD for  $v = 25$ ,  $k = 6$ .

**Generator 3.6:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+10$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 4u+1$ , replace  $\frac{5iK+6}{8}$  with  $v - \frac{5iK+6}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case II: If  $i = 4u+3$ , replace  $\frac{5iK+2}{8}$  with  $v - \frac{5iK+2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.6.**  $S = [1, 2, 3, 4, 5, 6, 9, 11, 14]$  provides minimal CNBD for  $v = 21$  and  $k = 10$ .

**Generator 3.7:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+7$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 8u+1$ , replace  $\frac{7iK+7}{8}$  with  $v - \frac{7iK+7}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ ,  $i > 1$  fork=7.

Case II: If  $i = 8u+3$ , replace  $\frac{3iK+1}{8}$  with  $v - \frac{3iK+1}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case III: If  $i = 8u+5$ , replace  $\frac{7iK+3}{8}$  with  $v - \frac{7iK+3}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case IV: If  $i = 8u+7$ , replace  $\frac{3iK+5}{8}$  with  $v - \frac{3iK+5}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 3.7.**  $S = [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 16, 17]$  provides minimal CNBD for  $v = 31$  and  $k = 15$ .

**Generator 3.8:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 4l+7$ ;  $l$  &  $u$  (integer)  $\geq 0$ .

Case I: If  $i = 8u+2$ , replace  $\frac{5iK+2}{8}$  with  $v - \frac{5iK+2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case II: If  $i = 8u+4$ , replace  $\frac{iK+4}{8}$  with  $v - \frac{iK+4}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case III: If  $i = 8u+6$ , replace  $\frac{5iK+6}{8}$  with  $v - \frac{5iK+6}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case IV: If  $i = 8u+8$ , replace  $\frac{iK}{8}$  with  $v - \frac{iK}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.8.**  $S_1 = [1, 2, 3, 4, 5, 8]$ ,  $S_2 = [9, 10, 11, 13, 14, 20]$  provide minimal CNBD for  $v = 29$  and  $k = 7$ .

**Generator 3.9:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+11$ ;  $l \& u(\text{integer}) \geq 0$ .

Case I: If  $i = 8u+1$ , replace  $\frac{7iK+3}{8}$  with  $v - \frac{7iK+3}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

Case II: If  $i = 8u+3$ , replace  $\frac{3iK+5}{8}$  with  $v - \frac{3iK+5}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case III: If  $i = 8u+5$ , replace  $\frac{7iK+7}{8}$  with  $v - \frac{7iK+7}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case IV: If  $i = 8u+7$ , replace  $\frac{3iK+1}{8}$  with  $v - \frac{3iK+1}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.9.**  $S = [1, 2, 3, 4, 5, 6, 7, 9, 11, 13]$  provides minimal CNBD for  $v = 23$  and  $k = 11$ .

**Generator 3.10:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l+12$ ;  $l \& u \geq 0$ .

Case I: If  $i$  is odd, replace  $\frac{iK+4}{8}$  with  $v - \frac{iK+4}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

Case II: If  $i$  is even, replace  $\frac{iK}{8}$  with  $v - \frac{iK}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.10.**  $S = [1, 3, 4, 5, 6, 7, 8, 10, 11, 13, 23]$  provides minimal CNBD for  $v = 25$ ,  $k = 12$ .

**Generator 3.11:** Minimal CNBDs can be obtained for  $v = 2ik+1$ ,  $k = 8l$ ;  $l \& u(\text{integer}) \geq 0$ .

- If  $i \geq 1$ , replace  $\frac{iK}{8}$  with  $v - \frac{iK}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 3.11.**  $S = [2, 3, 5, 6, 7, 8, 16]$  provides minimal CNBD for  $v = 17$  and  $k = 8$ .

#### 4. Minimal CNBDs in Blocks of Two Different Sizes

In this Section, some generators are developed to obtain minimal CNBDs in blocks of two different sizes through  $i+1$  sets of shifts. These  $i+1$  sets of shifts are obtained as follows. In all constructions,  $m = (v-1)/2$ .

- Divide the resultant elements of  $A$  (for required case of generator) into  $i$  groups of size  $k_1$  and one group of size  $k_2$  such that the sum of each group is divisible by  $v$ .
- By deleting one element (any) from each group, we will get the required  $i+1$  sets of shifts.

#### 4.1 Minimal CNBDs in blocks of two different sizes when $k_2 = 3$

**Generator 4.1.1:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 5 + c$ ,  $k_2 = 3 + c$ ,  $l$  &  $u$  (integer)  $\geq 0$ , where  $c = 0, 4$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{ik_1 + k_2}{8}$  with  $v - \frac{ik_1 + k_2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{5ik_1 + (5k_2 + 6)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 6)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{ik_1 + (k_2 + 4)}{8}$  with  $v - \frac{ik_1 + (k_2 + 4)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{5ik_1 + (5k_2 + 2)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 2)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.1.** Minimal CNBD is constructed for  $v = 35$ ,  $k_1 = 7$  and  $k_2 = 3$  from the following three sets of shifts.

$$S_1 = [1, 2, 3, 13, 18, 28], \quad S_2 = [6, 8, 9, 11, 12, 14], \quad S_3 = [4, 16]$$

**Generator 4.1.2:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 5$ ;  $k_2 = 3$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 2$ ; replacing  $\frac{3ik_1 + 10}{8}$  with  $v - \frac{3ik_1 + 10}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 4$ ; replacing  $\frac{7ik_1 + 28}{8}$  with  $v - \frac{7ik_1 + 28}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 6$ ; replacing  $\frac{3ik_1 + 14}{8}$  with  $v - \frac{3ik_1 + 14}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 8$ ; replacing  $\frac{7ik_1 + 24}{8}$  with  $v - \frac{7ik_1 + 24}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.2.** Minimal CNBD is constructed for  $v = 27$ ,  $k_1 = 5$  and  $k_2 = 3$  from the following three sets of shifts.

$$S_1 = [1, 2, 8, 13], \quad S_2 = [6, 7, 10, 22], \quad S_3 = [4, 12]$$

**Generator 4.1.3:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 9 + c$ ;  $k_2 = 3 + c$ , where  $c = 0, 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{ik_1 + (k_2 + 4)}{8}$  with  $v - \frac{ik_1 + (k_2 + 4)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{5ik_1 + (5k_2 + 2)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 2)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{ik_1 + k_2}{8}$  with  $v - \frac{ik_1 + k_2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{5ik_1 + (5k_2 + 6)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 6)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.1.3.** Minimal CNBD is constructed for  $v = 25$ ,  $k_1 = 9$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1,3,5,7,8,9,13,23], \quad S_2 = [4,11]$$

**Generator 4.1.4:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 6 + c$ ;  $k_2 = 3 + c$  where  $c = 0, 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 1$ ; replacing  $\frac{3ik_1 + (3k_2 + 5)}{8}$  with  $v - \frac{3ik_1 + (3k_2 + 5)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u + 3$ ; replacing  $\frac{3ik_1 + (3k_2 + 1)}{8}$  with  $v - \frac{3ik_1 + (3k_2 + 1)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.4.** Minimal CNBD is constructed for  $v = 19$ ,  $k_1 = 6$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1,2,6,8,15], \quad S_2 = [3,10]$$

**Generator 4.1.5:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 6$ ;  $k_2 = 3$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 2$ ; replacing  $\frac{7ik_1 + 28}{8}$  with  $v - \frac{7ik_1 + 28}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u + 4$ ; replacing  $\frac{7ik_1 + 24}{8}$  with  $v - \frac{7ik_1 + 24}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.5.** Minimal CNBD is constructed for  $v = 31$ ,  $k_1 = 6$  and  $k_2 = 3$  from the following three sets of shifts.

$$S_1 = [2,3,5,6,11], \quad S_2 = [7,8,9,12,16], \quad S_3 = [1,17]$$

**Generator 4.1.6:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 10 + c$ ;  $k_2 = 3 + c$  where  $c = 0, 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 1$ ; replacing  $\frac{3ik_1 + (3k_2 + 1)}{8}$  with  $v - \frac{3ik_1 + (3k_2 + 1)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u + 3$ ; replacing  $\frac{3ik_1 + (3k_2 + 5)}{8}$  with  $v - \frac{3ik_1 + (3k_2 + 5)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.1.6.** Minimal CNBD is constructed for  $v = 27$ ,  $k_1 = 6$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1,2,3,6,7,9,10,13,22], \quad S_2 = [4,12]$$

**Generator 4.1.7:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 7 + c$ ;  $k_2 = 3 + c$  where  $c = 0, 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{5ik_1 + (5k_2 + 6)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 6)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{ik_1 + k_2}{8}$  with  $v - \frac{ik_1 + k_2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{5ik_1 + (5k_2 + 2)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 2)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{ik_1 + (k_2 + 4)}{8}$  with  $v - \frac{ik_1 + (k_2 + 4)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.1.7.** Minimal CNBD is constructed for  $v = 21$ ,  $k_1 = 7$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1, 2, 3, 5, 6, 14], \quad S_2 = [4, 9]$$

**Generator 4.1.8:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 7$ ;  $k_2 = 3$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 2$ ; replacing  $\frac{3ik_1 + 14}{8}$  with  $v - \frac{3ik_1 + 14}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 4$ ; replacing  $\frac{7ik_1 + 28}{8}$  with  $v - \frac{7ik_1 + 28}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 6$ ; replacing  $\frac{3ik_1 + 10}{8}$  with  $v - \frac{3ik_1 + 10}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 8$ ; replacing  $\frac{7ik_1 + 24}{8}$  with  $v - \frac{7ik_1 + 24}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.8.** Minimal CNBD is constructed for  $v = 35$ ,  $k_1 = 7$  and  $k_2 = 3$  from the following sets of shifts.

$$S_1 = [1, 2, 3, 13, 18, 28], \quad S_2 = [6, 8, 9, 11, 12, 14], \quad S_3 = [4, 16]$$

**Generator 4.1.9:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 11 + c$ ;  $k_2 = 3 + c$  where  $c = 0, 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{5ik_1 + (5k_2 + 2)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 2)}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{ik_1 + (k_2 + 4)}{8}$  with  $v - \frac{ik_1 + (k_2 + 4)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{5ik_1 + (5k_2 + 6)}{8}$  with  $v - \frac{5ik_1 + (5k_2 + 6)}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{ik_1 + k_2}{8}$  with  $v - \frac{ik_1 + k_2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.9.** Minimal CNBD is constructed for  $v = 29$ ,  $k_1 = 11$  and  $k_2 = 3$  from the following two sets of shifts.



$$S_1 = [1,2,3,6,7,8,10,11,14,20], \quad S_2 = [4,13]$$

**Generator 4.1.10:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l$ ;  $k_2 = 3 + c$  where  $c = 0, 4$ ,  $i, l$  &  $u$  (integer)  $\geq 0$  by replacing  $\frac{7ik_1 + 8k_2}{8}$  with  $v - \frac{7ik_1 + 8k_2}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.10.** Minimal CNBD is constructed for  $v = 23$ ,  $k_1 = 8$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1,2,5,6,7,8,13], \quad S_2 = [3,11]$$

**Generator 4.1.11:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 4$ ;  $k_2 = 3$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i$  odd; replacing  $\{\frac{7}{2}(3i + 1)\}$  with  $v - \{\frac{7}{2}(3i + 1)\}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i$  even; replacing  $\{\frac{7}{2}(3i) + 3\}$  with  $v - \{\frac{7}{2}(3i) + 3\}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.1.11.** Minimal CNBD is constructed for  $v = 31$ ,  $k_1 = 12$  and  $k_2 = 3$  from the following two sets of shifts.

$$S_1 = [1,3,4,5,6,7,9,10,11,12,17], \quad S_2 = [2,16]$$

## 4.2 Minimal CNBDs in blocks of two different sizes when $k_2 = 4$

**Generator 4.2.1:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 5$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{3ik_1 + 17}{8}$  with  $v - \frac{3ik_1 + 17}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{7ik_1 + 31}{8}$  with  $v - \frac{7ik_1 + 31}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{3ik_1 + 13}{8}$  with  $v - \frac{3ik_1 + 13}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{7ik_1 + 35}{8}$  with  $v - \frac{7ik_1 + 35}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.1.** Minimal CNBD is constructed for  $v = 19$ ,  $k_1 = 5$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2,8,10,15], \quad S_2 = [1,5,7]$$

**Generator 4.2.2:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 5$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 2$ ; replacing  $\frac{5ik_1 + 22}{8}$  with  $v - \frac{5ik_1 + 22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 4$ ; replacing  $\frac{ik_1 + 4}{8}$  with  $v - \frac{ik_1 + 4}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 6$ ; replacing  $\frac{5ik_1 + 26}{8}$  with  $v - \frac{5ik_1 + 26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 8$ ; replacing  $\frac{ik_1 + 8}{8}$  with  $v - \frac{ik_1 + 8}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.2.** Minimal CNBD is constructed for  $v = 29$ ,  $k_1 = 5$  and  $k_2 = 4$  from the following three sets of shifts.

$$S_1 = [2, 3, 7, 13], \quad S_2 = [6, 8, 10, 20], \quad S_3 = [1, 5, 12]$$

**Generator 4.2.3:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 9$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{3ik_1 + 13}{8}$  with  $v - \frac{3ik_1 + 13}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{7ik_1 + 35}{8}$  with  $v - \frac{7ik_1 + 35}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{3ik_1 + 17}{8}$  with  $v - \frac{3ik_1 + 17}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{7ik_1 + 31}{8}$  with  $v - \frac{7ik_1 + 31}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.2.3.** Minimal CNBD is constructed for  $v = 27$ ,  $k_1 = 9$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2, 3, 6, 7, 8, 11, 13, 22], \quad S_2 = [1, 10, 12]$$

**Generator 4.2.4:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 6$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 1$ ; replacing  $\frac{5ik_1 + 26}{8}$  with  $v - \frac{5ik_1 + 26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u + 3$ ; replacing  $\frac{5ik_1 + 22}{8}$  with  $v - \frac{5ik_1 + 22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.2.4.** Minimal CNBD is constructed for  $v = 21$ ,  $k_1 = 6$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2, 4, 6, 11, 14], \quad S_2 = [1, 3, 9]$$

**Generator 4.2.5:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 6$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 2$ ; replacing  $\frac{ik_1 + 4}{8}$  with  $v - \frac{ik_1 + 4}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u + 4$ ; replacing  $\frac{ik_1 + 8}{8}$  with  $v - \frac{ik_1 + 8}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.5.** Minimal CNBD is constructed for  $v = 33$ ,  $k_1 = 6$  and  $k_2 = 4$  from the following three sets of shifts.

$$S_1 = [3, 4, 6, 15, 31], \quad S_2 = [8, 9, 10, 12, 16], \quad S_3 = [1, 5, 14]$$

**Generator 4.2.6:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 10$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 4u + 1$ ; replacing  $\frac{5ik_1 + 22}{8}$  with  $v - \frac{5ik_1 + 22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u + 3$ ; replacing  $\frac{5ik_1 + 26}{8}$  with  $v - \frac{5ik_1 + 26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.6.** Minimal CNBD is constructed for  $v = 29$ ,  $k_1 = 10$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2, 3, 4, 6, 7, 8, 13, 14, 20], \quad S_2 = [1, 5, 12]$$

**Generator 4.2.7:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 7$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{7ik_1 + 31}{8}$  with  $v - \frac{7ik_1 + 31}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{3ik_1 + 17}{8}$  with  $v - \frac{3ik_1 + 17}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{7ik_1 + 35}{8}$  with  $v - \frac{7ik_1 + 35}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{3ik_1 + 13}{8}$  with  $v - \frac{3ik_1 + 13}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.2.7.** Minimal CNBD is constructed for  $v = 23$ ,  $k_1 = 7$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2, 3, 4, 7, 11, 13], \quad S_2 = [1, 5, 9]$$

**Generator 4.2.8:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 7$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u+2$ ; replacing  $\frac{5ik_1+26}{8}$  with  $v - \frac{5ik_1+26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+4$ ; replacing  $\frac{ik_1+4}{8}$  with  $v - \frac{ik_1+4}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+6$ ; replacing  $\frac{5ik_1+22}{8}$  with  $v - \frac{5ik_1+22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+8$ ; replacing  $\frac{ik_1+8}{8}$  with  $v - \frac{ik_1+8}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.8.** Minimal CNBD is constructed for  $v = 37$ ,  $k_1 = 7$  and  $k_2 = 4$  from the following three sets of shifts.

$$S_1 = [3,4,6,14,17,19], \quad S_2 = [2,7,8,10,13,25], \quad S_3 = [1,5,16]$$

**Generator 4.2.9:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+11$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i = 8u+1$ ; replacing  $\frac{7ik_1+35}{8}$  with  $v - \frac{7ik_1+35}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+3$ ; replacing  $\frac{3ik_1+13}{8}$  with  $v - \frac{3ik_1+13}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+5$ ; replacing  $\frac{7ik_1+31}{8}$  with  $v - \frac{7ik_1+31}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+7$ ; replacing  $\frac{3ik_1+17}{8}$  with  $v - \frac{3ik_1+17}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.9.** Minimal CNBD is constructed for  $v = 31$ ,  $k_1 = 11$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [2,3,4,6,7,8,10,11,16,17], \quad S_2 = [1,5,13]$$

**Generator 4.2.10:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  if  $i > 0$  by replacing  $(i+1)$  with  $v-(i+1)$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.10.** Minimal CNBD is constructed for  $v = 25$ ,  $k_1 = 8$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [3,4,6,7,11,13,23], \quad S_2 = [1,9,10]$$

**Generator 4.2.11:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+12$ ;  $k_2 = 4$ ,  $l$  &  $u$  (integer)  $\geq 0$  for the following cases.

**Case I:**  $i$  odd; replacing  $\frac{ik_1+4}{8}$  with  $v - \frac{ik_1+4}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i$  even; replacing  $\frac{ik_1+8}{8}$  with  $v - \frac{ik_1+8}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.2.11.** Minimal CNBD is constructed for  $v = 33$ ,  $k_1 = 12$  and  $k_2 = 4$  from the following two sets of shifts.

$$S_1 = [3,4,6,7,8,9,10,11,15,16,31], \quad S_2 = [1,5,14]$$

### 4.3 Minimal CNBDs in blocks of two different sizes when $k_2 = 5$

**Generator 4.3.1:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 6$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u + 1$ ; replacing  $\frac{7ik_1 + 38}{8}$  with  $v - \frac{7ik_1 + 38}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u + 3$ ; replacing  $\frac{7ik_1 + 42}{8}$  with  $v - \frac{7ik_1 + 42}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.3.1.** Minimal CNBD Can be constructed for  $v = 23$ ,  $k_1 = 6$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,11,13], \quad S_2 = [1,2,7,8]$$

**Generator 4.3.2:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 6$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u + 2$ ; replacing  $\frac{3ik_1 + 20}{8}$  with  $v - \frac{3ik_1 + 20}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u + 4$ ; replacing  $\frac{3ik_1 + 16}{8}$  with  $v - \frac{3ik_1 + 16}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.2.** Minimal CNBD Can be constructed for  $v = 35$ ,  $k_1 = 6$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [4,6,15,16,18], \quad S_2 = [3,9,10,12,28], \quad S_3 = [1,5,13,14]$$

**Generator 4.3.3:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 10$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{7ik_1 + 42}{8}$  with  $v - \frac{7ik_1 + 42}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{7ik_1 + 38}{8}$  with  $v - \frac{7ik_1 + 38}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.3.** Minimal CNBD Can be constructed for  $v = 31$ ,  $k_1 = 10$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,7,8,10,13,16,17], \quad S_2 = [1,2,5,12].$$

**Generator 4.3.4:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 7$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{ik_1 + 9}{8}$  with  $v - \frac{ik_1 + 9}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{5ik_1 + 31}{8}$  with  $v - \frac{5ik_1 + 31}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{ik_1 + 5}{8}$  with  $v - \frac{ik_1 + 5}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{5ik_1 + 27}{8}$  with  $v - \frac{5ik_1 + 27}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.4.** Minimal CNBD can be constructed for  $v = 25$ ,  $k_1 = 7$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [5, 6, 7, 10, 13, 23], S_2 = [1, 3, 8, 9]$$

**Generator 4.3.5:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 7$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 2$ ; replacing  $\frac{7ik_1 + 38}{8}$  with  $v - \frac{7ik_1 + 38}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 4$ ; replacing  $\frac{3ik_1 + 20}{8}$  with  $v - \frac{3ik_1 + 20}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 6$ ; replacing  $\frac{7ik_1 + 42}{8}$  with  $v - \frac{7ik_1 + 42}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 8$ ; replacing  $\frac{3ik_1 + 16}{8}$  with  $v - \frac{3ik_1 + 16}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.5.** Minimal CNBD can be constructed for  $v = 39$ ,  $k_1 = 7$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3, 4, 6, 10, 19, 22], S_2 = [7, 8, 9, 11, 13, 18], S_3 = [1, 2, 5, 16]$$

**Generator 4.3.6:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 11$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{ik_1 + 5}{8}$  with  $v - \frac{ik_1 + 5}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{5ik_1 + 27}{8}$  with  $v - \frac{5ik_1 + 27}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{ik_1 + 9}{8}$  with  $v - \frac{ik_1 + 9}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{5ik_1 + 31}{8}$  with  $v - \frac{5ik_1 + 31}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.3.6.** Minimal CNBD can be constructed for  $v = 33$ ,  $k_1 = 11$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [5,6,7,8,9,10,14,15,16,31], \quad S_2 = [1,3,4,13]$$

**Generator 4.3.7:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$  if  $i > 0$  by replacing  $\frac{3ik_1+16}{8}$  with  $v - \frac{3ik_1+16}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.7.** Minimal CNBD can be constructed for  $v = 27$ ,  $k_1 = 8$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,6,7,8,12,13,22], \quad S_2 = [1,2,4,11]$$

**Generator 4.3.8:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 12$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i$  odd; replacing  $\frac{3ik_1+20}{8}$  with  $v - \frac{3ik_1+20}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i$  even; replacing  $\frac{3ik_1+16}{8}$  with  $v - \frac{3ik_1+16}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.8.** Minimal CNBD can be constructed for  $v = 35$ ,  $k_1 = 12$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,8,9,11,12,15,16,18,28], \quad S_2 = [1,2,5,14] \text{ provides minimal CNBD for}$$

**Generator 4.3.9:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 9$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{5ik_1+27}{8}$  with  $v - \frac{5ik_1+27}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{ik_1+5}{8}$  with  $v - \frac{ik_1+5}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{5ik_1+31}{8}$  with  $v - \frac{5ik_1+31}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{ik_1+9}{8}$  with  $v - \frac{ik_1+9}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.3.9.** Minimal CNBD can be constructed for  $v = 29$ ,  $k_1 = 9$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,7,8,12,14,20], \quad S_2 = [1,2,5,11]$$

**Generator 4.3.10:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 9$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+2$ ; replacing  $\frac{7ik_1+42}{8}$  with  $v - \frac{7ik_1+42}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+4$ ; replacing  $\frac{3ik_1+20}{8}$  with  $v - \frac{3ik_1+20}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u+6$ ; replacing  $\frac{7ik_1+38}{8}$  with  $v - \frac{7ik_1+38}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+8$ ; replacing  $\frac{3ik_1+16}{8}$  with  $v - \frac{3ik_1+16}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.10.** Minimal CNBD can be constructed for  $v = 47$ ,  $k_1 = 9$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,7,8,9,11,24], \quad S_2 = [10,12,13,14,16,17,18,26], \quad S_3 = [1,2,5,20]$$

**Generator 4.3.11:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+13$ ;  $k_2 = 5$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+1$ ; replacing  $\frac{5ik_1+31}{8}$  with  $v - \frac{5ik_1+31}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+3$ ; replacing  $\frac{ik_1+9}{8}$  with  $v - \frac{ik_1+9}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u+5$ ; replacing  $\frac{5ik_1+27}{8}$  with  $v - \frac{5ik_1+27}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+7$ ; replacing  $\frac{ik_1+5}{8}$  with  $v - \frac{ik_1+5}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.3.11.** Minimal CNBD can be constructed for  $v = 37$ ,  $k_1 = 13$  and  $k_2 = 5$  from the following sets of shifts.

$$S_1 = [3,4,6,7,8,9,10,11,16,17,19,25], \quad S_2 = [1,2,5,15]$$

#### 4.4 Minimal CNBDs in blocks of two different sizes when $k_2 = 6$

**Generator 4.4.1:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+7$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+1$ ; replacing  $\frac{3ik_1+19}{8}$  with  $v - \frac{3ik_1+19}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u+3$ ; replacing  $\frac{7ik_1+45}{8}$  with  $v - \frac{7ik_1+45}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+5$ ; replacing  $\frac{3ik_1+23}{8}$  with  $v - \frac{3ik_1+23}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u+7$ ; replacing  $\frac{7ik_1+49}{8}$  with  $v - \frac{7ik_1+49}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.4.1.** Minimal CNBD can be constructed for  $v = 27$ ,  $k_1 = 7$  and  $k_2 = 6$  from the following sets of shifts.



$$S_1 = [1,2,3,6,13,22], \quad S_2 = [4,8,9,11,12]$$

**Generator 4.4.2:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 4l + 7$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 2$ ; replacing  $\frac{ik_1 + 10}{8}$  with  $v - \frac{ik_1 + 10}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 4$ ; replacing  $\frac{5ik_1 + 36}{8}$  with  $v - \frac{5ik_1 + 36}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 6$ ; replacing  $\frac{ik_1 + 6}{8}$  with  $v - \frac{ik_1 + 6}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 8$ ; replacing  $\frac{5ik_1 + 32}{8}$  with  $v - \frac{5ik_1 + 32}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.2.** Minimal CNBD can be constructed for  $v = 41$ ,  $k_1 = 7$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,4,5,6,9,38], \quad S_2 = [7,8,10,11,13,21], \quad S_3 = [1,14,15,17,19]$$

**Generator 4.4.3:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 11$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u + 1$ ; replacing  $\frac{3ik_1 + 23}{8}$  with  $v - \frac{3ik_1 + 23}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u + 3$ ; replacing  $\frac{7ik_1 + 49}{8}$  with  $v - \frac{7ik_1 + 49}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u + 5$ ; replacing  $\frac{3ik_1 + 19}{8}$  with  $v - \frac{3ik_1 + 19}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u + 7$ ; replacing  $\frac{7ik_1 + 45}{8}$  with  $v - \frac{7ik_1 + 45}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.3.** Minimal CNBD can be constructed for  $v = 35$ ,  $k_1 = 11$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,3,4,5,6,9,10,12,18,28], \quad S_2 = [1,11,13,15,16]$$

**Generator 4.4.4:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$  if  $i > 0$  by replacing  $\frac{5ik_1 + 32}{8}$  with  $v - \frac{5ik_1 + 32}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.4.** Minimal CNBD can be constructed for  $v = 29$ ,  $k_1 = 8$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,3,5,7,8,13,14], \quad S_2 = [1,4,10,12,20]$$

**Generator 4.4.5:** Minimal CNBDs can be obtained for  $v = 2ik_1 + 2k_2 + 1$ ,  $k_1 = 8l + 12$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i$  odd; replacing  $\frac{5ik_1+36}{8}$  with  $v - \frac{5ik_1+36}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i$  even; replacing  $\frac{5ik_1+32}{8}$  with  $v - \frac{5ik_1+32}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.5.** Minimal CNBD can be constructed for  $v = 37$ ,  $k_1 = 12$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,3,4,5,6,8,9,10,13,19,25], \quad S_2 = [1,11,15,16,17]$$

**Generator 4.4.6:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+9$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+1$ ; replacing  $\frac{7ik_1+49}{8}$  with  $v - \frac{7ik_1+49}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+3$ ; replacing  $\frac{3ik_1+23}{8}$  with  $v - \frac{3ik_1+23}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u+5$ ; replacing  $\frac{7ik_1+45}{8}$  with  $v - \frac{7ik_1+45}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+7$ ; replacing  $\frac{3ik_1+19}{8}$  with  $v - \frac{3ik_1+19}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.6.** Minimal CNBD can be constructed for  $v = 31$ ,  $k_1 = 9$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,3,4,5,6,7,10,16], \quad S_2 = [1,8,11,13,17] \text{ provides minimal CNBD for.}$$

**Generator 4.4.7:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 4l+9$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+2$ ; replacing  $\frac{ik_1+6}{8}$  with  $v - \frac{ik_1+6}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u+4$ ; replacing  $\frac{5ik_1+36}{8}$  with  $v - \frac{5ik_1+36}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case III:**  $i = 8u+6$ ; replacing  $\frac{ik_1+10}{8}$  with  $v - \frac{ik_1+10}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u+8$ ; replacing  $\frac{5ik_1+32}{8}$  with  $v - \frac{5ik_1+32}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.7.** Minimal CNBD can be constructed for  $v = 49$ ,  $k_1 = 9$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,4,5,7,8,9,11,46], \quad S_2 = [10,12,14,16,17,18,23,24], \quad S_3 = [1,15,20,21,22]$$

**Generator 4.4.8:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+13$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+1$ ; replacing  $\frac{7ik_1+45}{8}$  with  $v - \frac{7ik_1+45}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u+3$ ; replacing  $\frac{3ik_1+19}{8}$  with  $v - \frac{3ik_1+19}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+5$ ; replacing  $\frac{7ik_1+49}{8}$  with  $v - \frac{7ik_1+49}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u+7$ ; replacing  $\frac{3ik_1+23}{8}$  with  $v - \frac{3ik_1+23}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.4.8.** Minimal CNBD can be constructed for  $v = 39$ ,  $k_1 = 13$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,3,5,6,7,8,9,11,12,13,18,19], \quad S_2 = [1,10,14,16,22]$$

**Generator 4.4.9:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+10$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u+1$ ; replacing  $\frac{ik_1+6}{8}$  with  $v - \frac{ik_1+6}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u+3$ ; replacing  $\frac{ik_1+10}{8}$  with  $v - \frac{ik_1+10}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.4.9.** Minimal CNBD can be constructed for  $v = 33$ ,  $k_1 = 10$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [3,4,5,6,7,9,10,16,31], \quad S_2 = [1,11,12,14,15]$$

**Generator 4.4.10:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 4l+10$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u+2$ ; replacing  $\frac{5ik_1+36}{8}$  with  $v - \frac{5ik_1+36}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u+4$ ; replacing  $\frac{5ik_1+32}{8}$  with  $v - \frac{5ik_1+32}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.10.** Minimal CNBD can be constructed for  $v = 53$ ,  $k_1 = 10$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [3,4,5,6,7,9,12,25,27], \quad S_2 = [2,10,11,13,16,18,19,20,36], \quad S_3 = [1,15,21,23,24]$$

**Generator 4.4.11:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+14$ ;  $k_2 = 6$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u+1$ ; replacing  $\frac{ik_1+10}{8}$  with  $v - \frac{ik_1+10}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 4u+3$ ; replacing  $\frac{ik_1+6}{8}$  with  $v - \frac{ik_1+6}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Example 4.4.11.** Minimal CNBD can be constructed for  $v = 41$ ,  $k_1 = 14$  and  $k_2 = 6$  from the following sets of shifts.

$$S_1 = [2,4,5,6,7,8,9,10,11,12,13,21,38], \quad S_2 = [1,14,15,17,19]$$

#### 4.5 Minimal CNBDs in blocks of two different sizes when $k_2 = 7$

**Generator 4.5.1:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 8l+12$ ;  $k_2 = 7$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i$  odd; replacing  $\frac{7ik_1+52}{8}$  with  $v - \frac{7ik_1+52}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i$  even; replacing  $\frac{7ik_1+56}{8}$  with  $v - \frac{7ik_1+56}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.5.1.** Minimal CNBD can be constructed for  $v = 39$ ,  $k_1 = 12$  and  $k_2 = 7$  from the following sets of shifts.

$$S_1 = [3,4,5,6,7,8,9,10,12,16,19], \quad S_2 = [1,2,11,14,15,22]$$

**Generator 4.5.2:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 4l+9$ ;  $k_2 = 7$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+2$ ; replacing  $\frac{3ik_1+26}{8}$  with  $v - \frac{3ik_1+26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case II:**  $i = 8u+4$ ; replacing  $\frac{7ik_1+52}{8}$  with  $v - \frac{7ik_1+52}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+6$ ; replacing  $\frac{3ik_1+22}{8}$  with  $v - \frac{3ik_1+22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case IV:**  $i = 8u+8$ ; replacing  $\frac{7ik_1+56}{8}$  with  $v - \frac{7ik_1+56}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.5.2.** Minimal CNBD can be constructed for  $v = 51$ ,  $k_1 = 9$  and  $k_2 = 7$  from the following sets of shifts.

$$S_1 = [4,5,7,8,9,13,24,26], \quad S_2 = [3,11,12,14,16,18,23,41], \quad S_3 = [1,2,17,19,21,22]$$

**Generator 4.5.3:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 4l+10$ ;  $k_2 = 7$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 4u+2$ ; replacing  $\frac{7ik_1+52}{8}$  with  $v - \frac{7ik_1+52}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 4u+4$ ; replacing  $\frac{7ik_1+56}{8}$  with  $v - \frac{7ik_1+56}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.5.3.** Minimal CNBD can be constructed for  $v = 55$ ,  $k_1 = 10$  and  $k_2 = 7$  from the following sets of shifts.

$$S_1 = [2,3,5,7,9,10,15,26,27], S_2 = [11,12,13,16,17,18,19,20,25], S_3 = [1,4,8,21,23,31]$$

**Generator 4.5.4:** Minimal CNBDs can be obtained for  $v = 2ik_1+2k_2+1$ ,  $k_1 = 4l+11$ ;  $k_2 = 7$ ,  $l$  &  $u$  (integer)  $\geq 0$ .

**Case I:**  $i = 8u+2$ ; replacing  $\frac{3ik_1+22}{8}$  with  $v - \frac{3ik_1+22}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case II:**  $i = 8u+4$ ; replacing  $\frac{7ik_1+52}{8}$  with  $v - \frac{7ik_1+52}{8}$  in  $A = [1, 2, \dots, m-1, m]$ .

**Case III:**  $i = 8u+6$ ; replacing  $\frac{3ik_1+26}{8}$  with  $v - \frac{3ik_1+26}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Case IV:**  $i = 8u+8$ ; replacing  $\frac{7ik_1+56}{8}$  with  $v - \frac{7ik_1+56}{8}$  in  $A = [1, 2, \dots, m-1, m+1]$ .

**Example 4.5.4.** Minimal CNBD can be constructed for  $v = 59$ ,  $k_1 = 11$  and  $k_2 = 7$  from the following sets of shifts.

$$S_1 = [4,5,6,7,8,9,12,28,29,48], \quad S_2 = [3,10,13,14,15,16,19,20,22,27], \\ S_3 = [1,2,17,23,24,26]$$



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