International Journal of Innovation, Creativity and Change. www.ijicc.net
Volume 12, Issue 5, 2020

The Bayesian Estimator for Probabilistic Dagum Distribution

Layla M.Nassir[a], Nathier A. Ibrahim[b], [a,b]College of Eng., Al-Mustansiriya University, Baghdad, Iraq, 2Al – Turath University, Email: [a]layla_matter@yahoo.com, [b]nbrhms@yahoo.com

The Dagum distribution has been widely used especially in the modelling of lifetime. It provides a statistical model that has a wide variety of application in many areas such as survival analysis and data analysis. The conventional maximum likelihood method is the usual way to estimate the parameters of a distribution. Bayesian approach has received much attention in contention with other estimation methods. In this paper we explore and compare the performance of the maximum likelihood and moment estimates with the Bayesian estimate of Survival function for Dagum distribution. The maximum likelihood estimation, moment estimation, Bayesian using Extension Jeffrey prior, gamma prior for estimation the Survival function under symmetric and asymmetric loss function are presented. We explore the performance of these estimators numerically under varying conditions. Through the simulation study a comparison are made on the performance of these estimators with respect to the integrated mean square error (MSE). For all the varying sample size, several values of the threshold parameter for the two parameter Dagum distribution and for the values for the extension of Jeffery and gamma prior. We concluded that Bayes method that based on Jeffery prior under squared error loss function are more efficient for small values of shape parameter and Bayes method that based on gamma under Linex loss function are more efficient for large values of shape parameter.

Key words: *Degum distribution, MSE, The maximum likelihood estimation, Linex.*

Introduction

Dagum distribution is a continuous distribution within the positive real numbers and was developed by Camilo Dagum in 1970 as a distribution of income data, and in recent years has been applied in many industrial and medical fields where it is used in life tests[1] and in atmospheric and other applications (Domma, F., Giordano, S. and Zenga, M. M. 2011) [3].

The probability density function of Dagum distribution with three parameters (α, β, ρ) is;

(1) $$0 < y < \infty, \alpha, \beta, \rho > 0 \qquad f(y;\alpha,\beta,\rho) = \frac{\rho\alpha y^{\alpha\rho-1}}{\beta^{\alpha\rho}\left[1+(y/\beta)^{\alpha}\right]^{\rho+1}}$$

Where ($:\alpha, \rho$) is the shape parameter, and ($:\beta$) is the scale parameter, then the CDF of this distribution is;

$$F(y;\alpha,\beta,\rho) = \Pr[Y \le y] = \int_0^y f(u)du$$

$$0 < y < \infty, \alpha, \beta, \rho > 0 \qquad = \left[1+(y/\beta)^{-\alpha}\right]^{-\rho} \qquad (2)$$

The survival function for this distribution is;

$$= 1-\left[1+(t/\beta)^{-\alpha}\right]^{-\rho} \qquad S(t) = \Pr\left[T > t\right] = 1 - F(t) = \int_t^{\infty} f(u;\alpha,\beta,\rho)du \qquad (3)$$

The risk function is;

$$h(t) = \frac{f(t)}{S(t)} = \frac{\dfrac{\rho\alpha y^{\alpha\rho-1}}{\beta^{\alpha\rho}\left[1+(t/\beta)^{\alpha}\right]^{\rho+1}}}{1-\left[1+(t/\beta)^{-\alpha}\right]^{-\rho}} \qquad (4)$$

From equation (4) we observe that failure rate is a variable over time (Shahzad, N, M. and Asghar, Z.;2013: 2015).

Figure 1. Drawing of distribution function for different values of parameters distribution

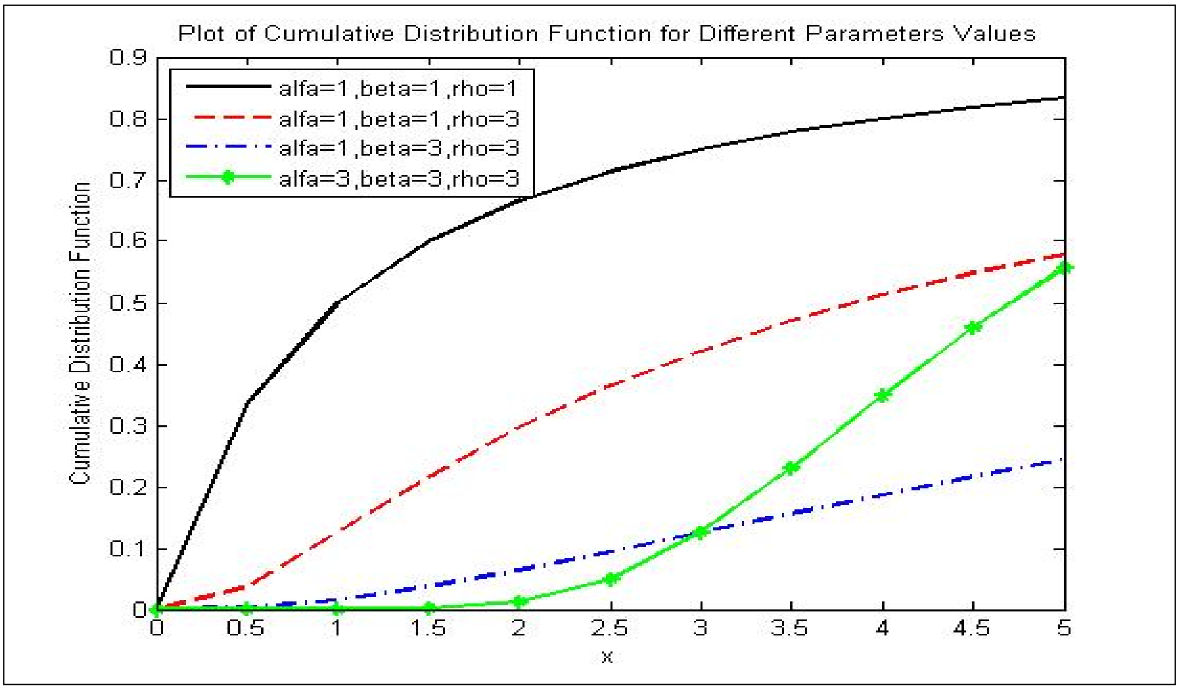


Figure 2. p.d.f for different values of parameters distribution

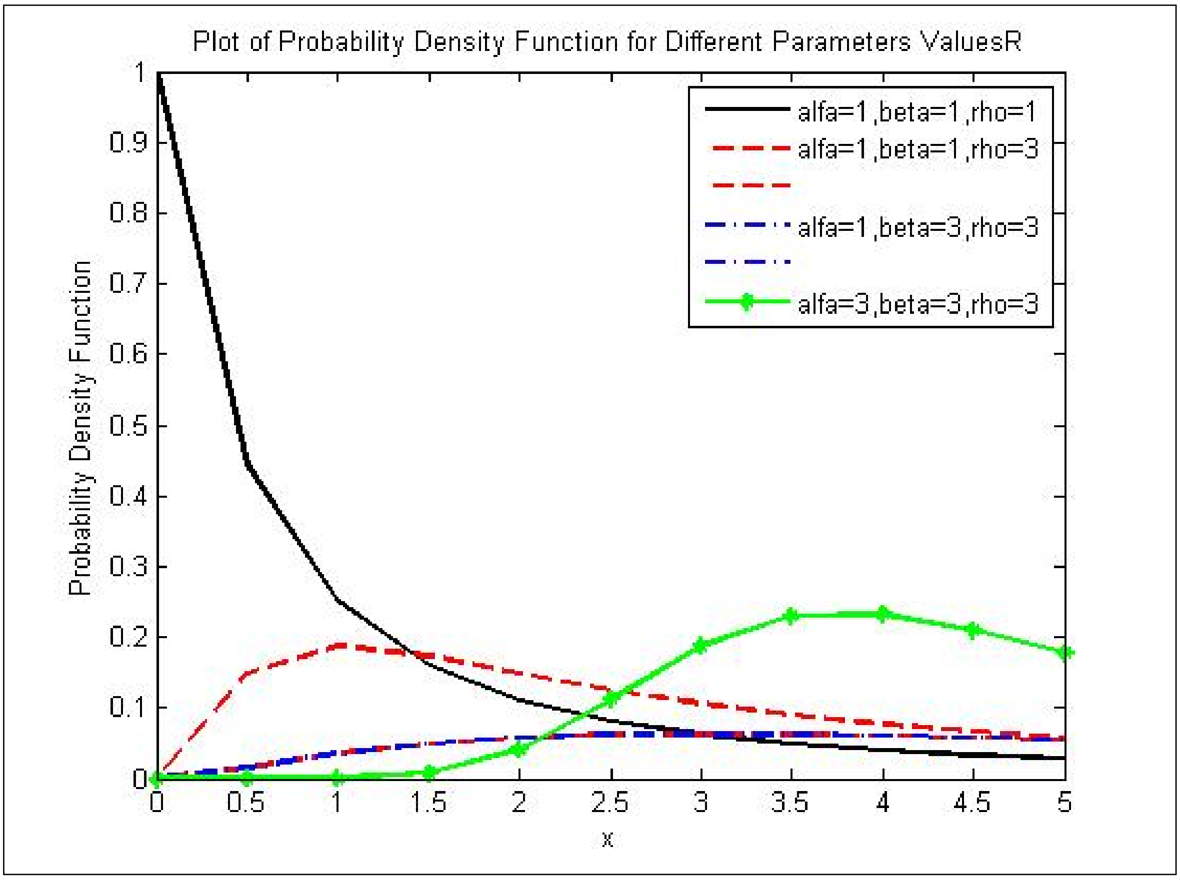
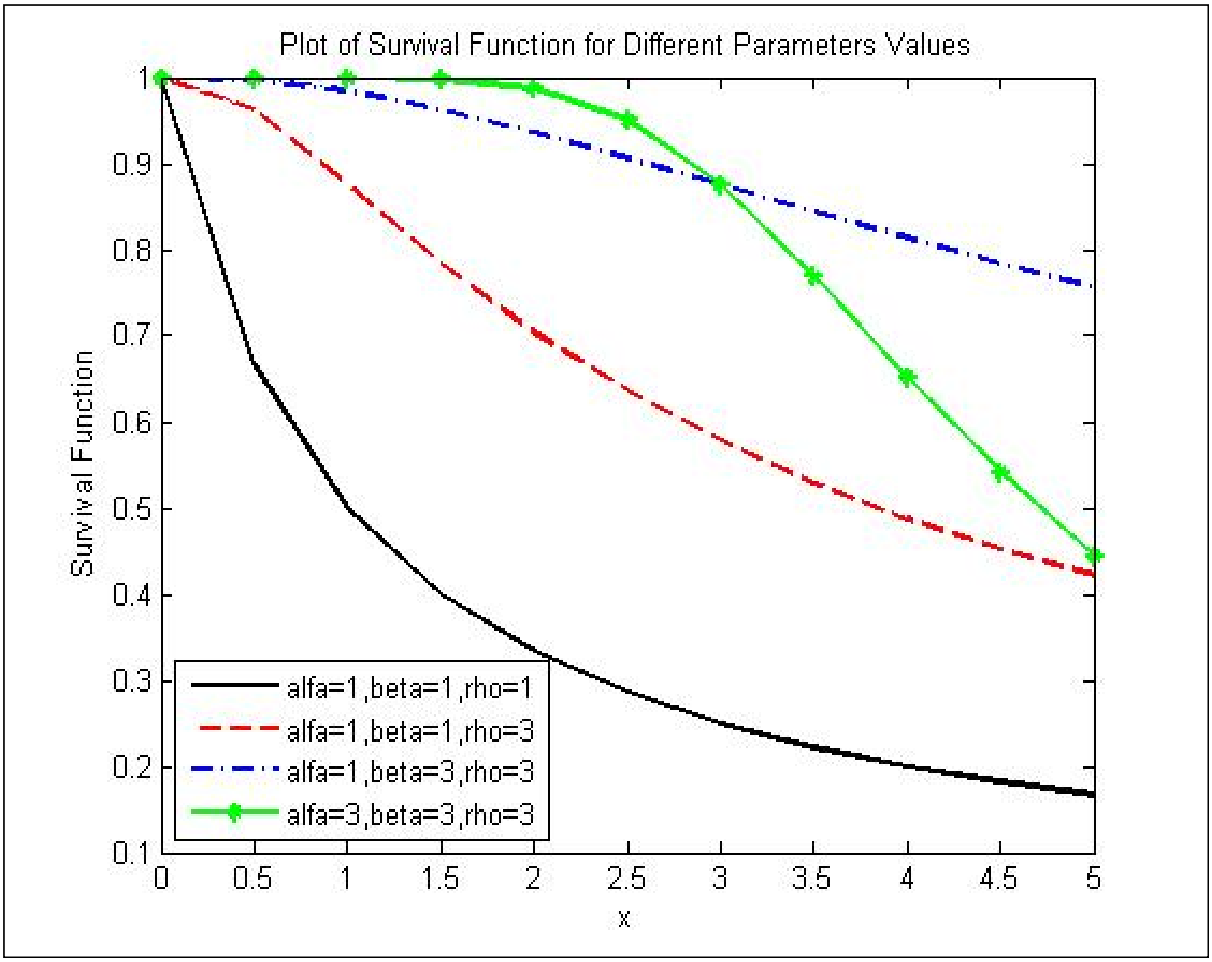


Figure 3. Survival function for different values of parameters distribution



In order to estimate the survival function, there were many traditional methods, like MLE, MOM, which has different statistical characteristics, but neglects the available initial information about the parameters to be estimated, for that a derivative to estimate survival function according to Bayesian methods which depend on initial information (Bander Al-Zahrani and Samerah Basloom, 2016); (Omari,M.A, Kutubi,H,S and Ibrahim,N,A 2010). The problem in the theoretical comparison between these methods since it is not possible to obtain the theoretical distribution of the capabilities of those methods which made it difficult to identify any better method based on the known statistical criteria (ALAA A, 2015).

The aim of the paper is to make an experimental comparison between the Bayesian methods, MLE, and moments, to find the best way to estimate the survival function of this distribution according to the statistical measurement (MSE).

Theoretical Aspect

The estimation methods for estimating shape parameter and survival function with fixed other parameters are;

Maximum Likelihood Method[4]

This method aims to make the function of the likelihood function at its maximum limit (BRODERICK O. OLUYEDE, SASITH RAJASOORIYA, 2013). If we have a random sample $(y_1, y_2, ..., y_n)$ that has a distribution of Dagum with three parameters (α, β, ρ), it is the maximum likelihood estimator that makes the function at its maximum limit and can be obtained by deriving the logarithm of the maximum likelihood function and equalling it to zero, if (y) distributed Dagum with three parameters (α, β, ρ), the maximum likelihood function will be as follows (Oluyede, B.O. and Rajasooriya, S 2013)..:

$$L(y_1, y_2, ..., y_n; \alpha, \beta, \rho) = \prod_{i=1}^{n} f(y_i; \alpha, \beta, \rho) = \rho^n \alpha^n \prod_{i=1}^{n} y_i^{\alpha\rho-1} \beta^{-n\alpha\rho} \prod_{i=1}^{n} \left(1 + (y_i/\beta)^{\alpha}\right)^{-\rho-1} \quad (5)$$

Taking logarithm for both sides;

$$\log(L) = n\log\rho + n\log(\alpha) + (\alpha\rho - 1)\sum_{i=1}^{n}\log(y_i) - n\alpha\rho\log(\beta) - (\rho+1)\sum_{i=1}^{n}\log\left(1 + (y_i/\beta)^{\alpha}\right) (6)$$

Assuming the fixing of (α, β) and derivative of (6) according to (ρ), we get;

$$\hat{\rho}_{MLE} = \frac{n}{\sum_{i=1}^{n}\left(\alpha\log\beta - \alpha\log y_i + \log(1 + (y_i/\beta)^{\alpha})\right)} \quad (7)$$

Depending on invariant property, (Saima Naqash, S. P. Ahmad and Aquil Ahmed, 2017) the survival function of maximum likelihood is;

$$\hat{S}_{MLE}(t) = 1 - [1 + (t/\beta)^{-\alpha}]^{-\hat{\rho}_{MLE}} \quad (8)$$

Moments Method

Moments method is an easy method, depend on the hypothesis of the equivalence between population moments (μ_n) and sample moments (m_n), then solution the equations to find parameters estimations, the moment function of Dagum distribution with three parameters α, β, ρ) is;

(9) $$\mu_r = E(y^r) = \rho\beta^r B(\rho + r/\alpha, 1 - r/\alpha)$$

Where;

$B(\cdot,\cdot)$: Beta function

By the equivalent of first sample moment and first population moment we get;

$$\bar{y} = \rho\beta B(\rho + 1/\alpha, 1 - 1/\alpha) \quad . \qquad (10)$$

Simply equation (10);

$$\bar{y} - \beta \frac{\Gamma(\rho + 1/\alpha)\Gamma(1 - 1/\alpha)}{\Gamma(\rho)} = 0 \qquad (11)$$

Solving equation (11) using frequency Newton Raphsin method, we get moment estimator ($\hat{\rho}_{MO}$) , then reliability function estimator is;

(12) $$\hat{S}_{MO}(t) = 1 - [1 + (t/\beta)^{-\alpha}]^{-\hat{\rho}_{MO}} \quad .$$

Bayesian Method

This method depend on extended Jeffrey initial information, (Sanku Dey, Bander Al-Zahrani & Samerah Basloom, 2017) let $(y_1, y_2, ..., y_n)$ be random sample distributed Dagum distribution with three parameters (α, β, ρ), then extended Jeffrey initial information can be calculated as;

(13) $$g_1(\rho) = k^c \sqrt{I^c(\rho)}$$

$I(\rho) = -E\left(\frac{\partial^2 \log L}{\partial \rho^2}\right)$ represents Fisher information, then;

(14) $$g_1(\rho) = k\frac{n^{c/2}}{\rho^c}$$

To get next distribution depend on Jeffrey initial information, we apply Bayes formula below;

(15) $$h(\rho \setminus y_1, y_2, ..., y_n) = \frac{L(y_1, y_2, ..., y_n; \rho) g_1(\rho)}{\int_0^{\infty} L(y_1, y_2, ..., y_n; \rho) g_1(\rho) d\rho}$$.

(16) $$\int_0^{\infty} L(y_1, y_2, ..., y_n; \rho) g_1(\rho) d\rho = \int_0^{\infty} k^c \frac{n^{c/2}}{\rho^c} \rho^n \alpha^n \prod_{i=1}^{n} t_i^{\alpha\rho-1} \beta^{-n\alpha\rho} \prod_{i=1}^{n} \left[1 + (y_i / \beta)^{\alpha}\right]^{-(\rho+1)} d\rho$$

After some mathematical procedures the integral of equation (16) will be;

$$\int_0^{\infty} L(y_1, y_2, ..., y_n; \rho) g_1(\rho) d\rho = \frac{k^c n^{c/2} \alpha^n \Gamma(n-c+1)}{e^{\sum_{i=1}^{n} \log y_i} e^{\sum_{i=1}^{n} \log\left[1+(y_i/\beta)^{\alpha}\right]} \left[\Sigma_1\right]^{n-c+1}} \tag{17}$$

Where;

$$\Sigma_1 = \sum_{i=1}^{n} \left(\alpha \log \beta - \alpha \log y_i + \log\left[1 + (y_i / \beta)^{\alpha}\right]\right)$$

$\Gamma(.)$: Gamma function

The next conditional p.d.f of (ρ) in present of sample data $(y_1, y_2, ... y_n)$ is;

(18) $$\rho > 0 \qquad h(\rho \setminus y_1, y_2, ..., y_n) = \frac{\left[\Sigma_1\right]^{n-c+1} \rho^{n-c} e^{-\rho\Sigma_1}}{\Gamma(n-c+1)}$$

That means the distribution of (ρ) parameter be Gamma distribution

[$\rho \sim Gamma\ (\Sigma_1\ \&\ n-c+1)$]

Depending on squared Error Loss Function which considered as common loss function, the Bayes estimator be the expected of next previous distribution according to Gamma distribution, the next expectation be (Lee, E.T. 2017);

(19) $$\hat{\rho}_{BJ} = \underset{post}{E}(\rho) = \int_0^{\infty} h(\rho) d\rho = \frac{n-c+1}{\Sigma_1} = \frac{n-c+1}{\sum_{i=1}^{n} (\alpha \log \beta - \alpha \log y_i + \log(1 + (y_i / \beta)^{\alpha})}$$

Also Bayes estimator for survival function depends on next distribution is;

(20) $\hat{S}(t)_{BJ} = \underset{post}{E}(S(t)) = \int_0^\infty \left(1 - \left[1 + (t/\beta)^{-\alpha}\right]^{-\rho}\right) h(\rho) d\rho = 1 - \left[\frac{1}{1 + \frac{\log(1 + (t/\beta)^{-\alpha})}{\Sigma_1}}\right]^{n-c+1}$

Depending on Symmetric Precautionary Error Loss Function, which considered as symmetric functions as;

(21) $$L(\rho, d) = \frac{(\rho - d)^2}{d}$$

Depending on previous loss function, Bayes estimates that make risk function in its minimum limit be;

(22) $$\hat{\rho} = \sqrt{E(\rho)^2}$$

According to the expected of previous next distribution, the previous expectation be;

$$\underset{post}{E}(\rho^2) = \int_0^\infty h(\rho) d\rho = \frac{(n - c + 2)(n - c + 1)}{\Sigma_1^2} \quad (23)$$

(24) $$\hat{\rho}_{BJP} = \frac{\sqrt{(n - c + 2)(n - c + 1)}}{\Sigma_1}$$

Also, Bayes estimator of survival function is the expected survival function depending on next distribution as well as previous loss function;

(25) $$\hat{S}(t)_{BJP} = \sqrt{\underset{post}{E}(S(t)^2)} = \sqrt{E\left[1 - 2(1 + (t/\beta)^{-\alpha})^{-\rho} + 2(1 + (t/\beta)^{-\alpha})^{-2\rho}\right]}$$

Finding the previous mathematical expectation and substituting in equation (25) to find survival function estimator as;

(26) $$\hat{S}(t)_{BJP} = \sqrt{1 - 2\left[\frac{1}{1 + \frac{\log(1 + (t/\beta)^{-\alpha})}{\Sigma_1}}\right]^{n-c+1} + \left[\frac{1}{1 + \frac{2\log(1 + (t/\beta)^{-\alpha})}{\Sigma_1}}\right]^{n-c+1}}$$

Depending on Asymmetric Linear Expo Error Loss Function, which gives biased unequal weighted for upper limit than lower limit as;

$$L(\rho,d)=\left[\frac{d}{\rho}-\log\frac{d}{\rho}-1\right] \qquad (27)$$

Bayes estimator makes risk function in minimum limit depending on previous loss function is;

$$\hat{\rho}=\left[E(\rho^{-1})\right]^{-1} \qquad (28)$$

Previous expectation according to next expectation be;

$$\underset{post}{E}(\rho^{-1})=\int_0^\infty h(\rho)d\rho=\frac{\Sigma_1}{n-c} \qquad (29)$$

$$\hat{\rho}_{BJL}=\frac{n-c}{\Sigma_1} \qquad (30)$$

Also Bayes estimator for survival data is the expectation for survival function depending on next distribution and previous loss function as;

$$\hat{S}(t)_{BJL}=\left(\underset{post}{E}(S(t)^{-1})\right)^{-1}=\int_0^\infty\left(1-(1+(t/\beta)^{-\alpha})^{-\rho}\right)^{-1}\frac{\Sigma_1^{n-c+1}\rho^{n-c}e^{-\Sigma_1\rho}}{\Gamma(n-c+1)}d\rho \qquad (31)$$

Making use the following sequence;

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{j=0}^{\infty}x^j \qquad (32)$$

The previous integration be;

$$\hat{S}(t)_{BJL}=\left(\underset{post}{E}(S(t)^{-1})\right)^{-1}=\int_0^\infty\sum_{j=0}^{\infty}(1+(t/\beta)^{-\alpha})^{-j\rho}\frac{\Sigma_1^{n-c+1}\rho^{n-c}e^{-\Sigma_1\rho}}{\Gamma(n-c+1)}d\rho \qquad (33)$$

Finding the previous mathematical expectation and substituting in equation (33) we can find survival function estimator as;

$$\hat{S}(t)_{BJL} = \left(\sum_{j=0}^{\infty} \left[\frac{1}{1 + \dfrac{j \log(1 + (t/\beta)^{-\alpha})}{\Sigma_1}} \right]^{n-c+1} \right)^{-1} \quad (34)$$

Bayes Method with GPB

We consider the initial information as informational by Gamma distribution;

$$(35) \qquad \alpha_1, \beta_1, \rho > 0 \qquad g_2(\rho) = \frac{\alpha_1^{\beta_1} \rho^{\beta_1 - 1} e^{-\alpha_1 \rho}}{\Gamma \beta_1}$$

So the initial information submitting to previous distribution, and if $(y_1, y_2, ..., y_n)$ is a random sample distributed Dagum with three parameters (α, β, ρ), then the next distribution depending on Gamma information can be apply by Bayes formula below;

$$(36) \qquad h(\rho \setminus y_1, y_2, ..., y_n) = \frac{L(y_1, y_2, ..., y_n; \rho) g_2(\rho)}{\int_0^{\infty} L(y_1, y_2, ..., y_n; \rho) g_2(\rho) d\rho}$$

So;

$$\int_0^{\infty} L(y_1, y_2, ..., y_n; \rho) g_2(\rho) d\rho = \int_0^{\infty} \frac{\alpha_1^{\beta}{}_1 \rho^{\beta_1 - 1} e^{-\alpha_1 \rho}}{\Gamma \beta_1} \rho^n \alpha^n \prod_{i=1}^{n} t_i^{\alpha\rho - 1} \beta^{-n\alpha\rho} \prod_{i=1}^{n} \left[1 + (y_i / \beta)^{\alpha}\right]^{-(\rho+1)} d\rho \quad (37)$$

Where $\Gamma.$ is Gamma function

The next conditional p.d.f for (ρ) parameter in present of sample data $(y_1, y_2, ..., y_n)$, after some mathematical treatments;

$$(38) \qquad \rho > 0 \quad h(\rho \setminus y_1, y_2, ..., y_n) = \frac{[\Sigma_2]^{n+\beta_1} \rho^{n+\beta_1 - 1} e^{-\Sigma_2 \rho}}{\Gamma n + \beta_1}$$

Where;

$$\Sigma_2 = \alpha_1 + \sum_{i=1}^{n}\left(\alpha \log y_i - \alpha \log \beta + \log\left[1 + (y_i/\beta)^{\alpha}\right]\right)$$

Means that the distribution of the parameter (ρ) according to Gamma distribution with parameter $\rho \sim Gamma\ (\Sigma_2\ \&\ n+\beta_1)$.

Depending on Squared Error Loss Function, which considered as common loss function, the Bayes estimator for previous next distribution expectation is;

$$(39)\quad \hat{\rho}_{BG} = \underset{post}{E}(\rho) = \int_0^{\infty} \rho h(\rho) d\rho = \frac{n+\beta_1}{\Sigma_2} = \frac{n+\beta_1}{\alpha_1 + \sum_{i=1}^{n}\left(\alpha \log y_i - \alpha \log \beta + \log\left[1 + (y_i/\beta)^{\alpha}\right]\right)}$$

Also Bayes estimator for survival function is the survival function estimator depending on previous next distribution;

$$\hat{S}(t)_{BG} = \underset{post}{E}(S(t)) = \int_0^{\infty}\left(1 - \left[1 + (t/\beta)^{-\alpha}\right]^{-\rho}\right) h(\rho) d\rho = 1 - \left[\frac{1}{1 + \frac{\log(1 + (t/\beta)^{-\alpha})}{\Sigma_2}}\right]^{n+\beta_1} \quad (40)$$

Depending on Symmetric Precautionary Error Loss Function, which considered as symmetric function in the form;

$$(41)\qquad L(\rho, d) = \frac{(\rho - d)^2}{d}$$

Depending on previous loss function, Bayes estimator that makes risk function in its minimum limit is;

$$(42)\qquad \hat{\rho} = \sqrt{E(\rho)^2}$$

According to expectation of previous next distribution, the previous expectation is;

$$\underset{post}{E}(\rho^2) = \int_0^{\infty} h(\rho) d\rho = \frac{(n+\beta_1+1)(n+\beta_1)}{\Sigma_2^2} \quad (43)$$

$$\hat{\rho}_{BGP} = \frac{\sqrt{(n+\beta_1)(n+\beta_1+1)}}{\Sigma_2} \quad (44)$$

Also Bayes estimator of survival function is the expectation of survival function depending on next distribution and previous loss function as;

$$(45)\quad \hat{S}(t)_{BGP} = \sqrt{\underset{post}{E}(S(t)^2)} = \sqrt{E\left[1 - 2(1+(t/\beta)^{-\alpha})^{-\rho} + 2(1+(t/\beta)^{-\alpha})^{-2\rho}\right]}$$

Finding the previous mathematical expectation and substituting in (45) get survival function estimator as;

$$(46)\quad \hat{S}(t)_{BGP} = \sqrt{1 - 2\left[\frac{1}{1+\frac{\log(1+(t/\beta)^{-\alpha})}{\Sigma_2}}\right]^{n+\beta_1} + \left[\frac{1}{1+\frac{2\log(1+(t/\beta)^{-\alpha})}{\Sigma_2}}\right]^{n+\beta_1}}$$

Depending on Asymmetric Linear Expo Error Loss Function Linex, which considered as non – symmetric gives biased unequalled weighted for upper limits than lower limits take shape below;

$$(47)\quad L(\rho, d) = \left[\frac{d}{\rho} - \log\frac{d}{\rho} - 1\right]$$

Depending on previous loss function, Bayes estimator, which makes loss function in the minimum limit is;

$$(48)\quad \hat{\rho} = \left[E(\rho^{-1})\right]^{-1}$$

According to the expectation of previous next distribution, the previous expectation is;

$$(49)\quad \underset{post}{E}(\rho^{-1}) = \int_0^\infty h(\rho)d\rho = \frac{\Sigma_2}{n+\beta_1-1}$$

$$(50)\quad \hat{\rho}_{BGL} = \frac{n+\beta_1-1}{\Sigma_2}$$

Also Bayes estimator of survival function is the expectation of survival function according to next distribution and previous loss function as;

$$(51)\quad \hat{S}(t)_{BGL} = \left(\underset{post}{E}(S(t)^{-1})\right)^{-1} = \int_0^\infty \left(1-(1+(t/\beta)^{-\alpha})^{-\rho}\right)^{-1} \frac{\Sigma_2^{n+\beta_1}\rho^{n+\beta_1-1}e^{-\Sigma_2\rho}}{\Gamma(n+\beta_1)}d\rho$$

Using sequence below;

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j \tag{52}$$

The integral is;

$$\hat{S}(t)_{BGL} = \left(\underset{post}{E}(S(t)^{-1}) \right)^{-1} = \int_0^{\infty} \sum_{j=0}^{\infty} (1 + (t/\beta)^{-\alpha})^{-j\rho} \frac{\Sigma_2^{n+\beta_1} \rho^{n+\beta_1 - 1} e^{-\Sigma_2 \rho}}{)n+\beta_1} d\rho \tag{53}$$

Finding the previous mathematical expectation and substituting in (53) get survival function estimator as;

$$\hat{S}(t)_{BGL} = \left(\sum_{j=0}^{\infty} \left[\frac{1}{1 + \frac{\log(1 + (t/\beta)^{-j\alpha})}{\Sigma_2}} \right]^{n+\beta_1} \right)^{-1} \tag{54}$$

Experimental Aspect

We used simulation procedure for compression between different methods, that this procedure has been done through steps below;

a. Hypothesis values has been chosen for sample size (n =10, 15,25,50,100) represents small, medium, and large sample size, and two values for shape parameter ($\rho = 0.5,1$), also ($\alpha = 0.5,1$), ($\beta = 0.5,1$), ($c = 1,2$), and ($\alpha_1, \beta_1 = 1,2$), as shown in the following table;

Table 1: Hypothesis parameters values for simulation experiments

Case No.	α	β	ρ	c	α_1	β_1
1	0.5	0.5	0.5	1	1	1
2	0.5	0.5	0.5	2	1	1
3	0.5	0.5	0.5	1	2	2
4	0.5	0.5	0.5	2	2	2
5	0.5	0.5	1	1	1	1
6	0.5	0.5	1	2	1	1
7	0.5	0.5	1	1	2	2
8	0.5	0.5	1	2	2	2
9	0.5	1	1	1	1	1
10	0.5	1	1	2	1	1
11	1	1	1	1	1	1
12	1	1	1	2	1	1

b. Data generation according to Dagum distribution with three parameters (α, β, ρ) as;

$$U_i \sim U(0,1) \quad i = 1,\ldots,n \qquad (55)$$

$$F(y;\alpha,\beta,\rho) = \left[1 + (y/\beta)^{-\alpha}\right]^{-\rho} \qquad (56)$$

$$U = \left[1 + (y/\beta)^{-\alpha}\right]^{-\rho} \qquad (57)$$

$$y = \beta\left(U^{-1/\rho} - 1\right)^{-1/\alpha} \qquad (58)$$

c. Estimation of Dagum distribution parameters with (α, β, ρ) parameters for all methods and use it to estimate survival function depending on (t_i) generated from step (b) to get the best estimator using MSE as;

$$MSE(S^{\wedge}(t)) = \frac{1}{L}\sum_{i=1}^{L}\left((S_i^{\wedge}(t) - S_i^{\wedge}(t))^2\right) \qquad (59)$$

Where (L=1000) is the experiment frequency, and by using MATLAB-R2016a, the following tables shows the results.

Table 2: MSE for all methods and different sample size for first case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.004234	0.064204	0.004032	0.004056	0.003198	0.003687	0.003883	0.002915	BGL
	1.5	0.002442	0.031368	0.002347	0.002439	0.001517	0.00219	0.002376	0.001628	BJL
	2.5	0.001785	0.021582	0.001725	0.001815	0.00131	0.00162	0.001778	0.001506	BJL
	3.5	0.001427	0.016659	0.001383	0.001467	0.00137	0.001304	0.001441	0.001595	BJL
	4.5	0.001196	0.013647	0.001162	0.001239	0.001503	0.001099	0.00122	0.001739	BGS
15	0.5	0.004272	0.056191	0.003994	0.004348	0.003101	0.004154	0.004573	0.003285	BJL
	1.5	0.00243	0.027578	0.002317	0.002575	0.001991	0.002432	0.002727	0.002229	BJL
	2.5	0.001767	0.019007	0.001698	0.001902	0.001906	0.001788	0.002019	0.002143	BJS
	3.5	0.001407	0.014685	0.001359	0.001529	0.002001	0.001433	0.001625	0.002229	BJS
	4.5	0.001177	0.012038	0.001141	0.001287	0.002141	0.001204	0.001369	0.002361	BJS
25	0.5	0.00193	0.064122	0.001831	0.001987	0.001546	0.001946	0.002131	0.001671	BJL
	1.5	0.001095	0.03146	0.001056	0.001162	0.001231	0.001128	0.001251	0.001376	BJS
	2.5	0.000795	0.021678	0.000772	0.000854	0.001375	0.000826	0.00092	0.001519	B
	3.5	0.000633	0.016747	0.000617	0.000684	0.00158	0.00066	0.000738	0.001719	BJS
	4.5	0.000529	0.013727	0.000517	0.000575	0.001785	0.000554	0.00062	0.00192	BJS
50	0.5	0.000878	0.063764	0.000876	0.000867	0.000682	0.000853	0.000854	0.000686	BGL
	1.5	0.000478	0.031285	0.000477	0.000474	0.0005	0.000466	0.00047	0.000545	BGS
	2.5	0.000342	0.021558	0.000341	0.00034	0.0007	0.000334	0.000337	0.000754	BGS

	3.5	0.00027	0.016654	0.000269	0.000269	0.000944	0.000264	0.000267	0.001	BGS
	4.5	0.000224	0.013651	0.000224	0.000223	0.00118	0.00022	0.000222	0.001237	BGS
100	0.5	0.000954	0.058862	0.000936	0.000969	0.000948	0.000968	0.001003	0.000988	BJS
	1.5	0.000527	0.029022	0.000521	0.000542	0.000961	0.000539	0.000561	0.001	BJS
	2.5	0.000379	0.020033	0.000376	0.000391	0.001199	0.000389	0.000406	0.001237	BJS
	3.5	0.0003	0.015491	0.000298	0.00031	0.001447	0.000309	0.000322	0.001483	BJS
	4.5	0.00025	0.012706	0.000248	0.000259	0.001677	0.000257	0.000269	0.001711	BJS
Best										BJS

Table 3: MSE for all methods and different sample size for second case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.005832	0.042918	0.004684	0.004682	0.00373	0.005231	0.005837	0.004141	BJL
	1.5	0.003531	0.021138	0.002749	0.002848	0.001725	0.003269	0.003732	0.002778	BJL
	2.5	0.002628	0.014587	0.002027	0.00213	0.001427	0.002464	0.002838	0.002548	BJL
	3.5	0.002121	0.011278	0.001629	0.001726	0.001444	0.002003	0.00232	0.002553	BJL
	4.5	0.001791	0.009249	0.001371	0.001462	0.001553	0.0017	0.001976	0.002632	BJS
15	0.5	0.005748	0.056585	0.004715	0.004864	0.003689	0.005436	0.005857	0.004436	BJL
	1.5	0.003366	0.027822	0.002726	0.002872	0.001979	0.003258	0.00357	0.002876	BJL
	2.5	0.002474	0.019186	0.001997	0.002122	0.001731	0.002417	0.002666	0.002599	BJL
	3.5	0.001983	0.014828	0.001598	0.001706	0.001756	0.001948	0.002157	0.002584	BJS
	4.5	0.001666	0.012158	0.001342	0.001437	0.001863	0.001643	0.001824	0.002652	BJS

25	0.5	0.001367	0.05892	0.001557	0.001448	0.001112	0.001293	0.001297	0.000976	BGL
	1.5	0.000735	0.02899	0.000823	0.000768	0.000405	0.000701	0.000711	0.000609	BJL
	2.5	0.000523	0.019996	0.000583	0.000545	0.000485	0.000501	0.000511	0.000769	BJL
	3.5	0.000412	0.015457	0.000457	0.000428	0.000686	0.000395	0.000404	0.000996	BGS
	4.5	0.000342	0.012674	0.000378	0.000354	0.000905	0.000328	0.000336	0.001224	BGS
50	0.5	0.001281	0.062474	0.001133	0.00117	0.000992	0.001293	0.001359	0.001197	BJL
	1.5	0.000714	0.030719	0.000632	0.000657	0.000832	0.000729	0.000772	0.001059	BJS
	2.5	0.000515	0.021184	0.000456	0.000476	0.001029	0.000529	0.000561	0.001254	BJS
	3.5	0.000409	0.016372	0.000361	0.000378	0.001263	0.00042	0.000447	0.001483	BJS
	4.5	0.000341	0.013424	0.000302	0.000316	0.001489	0.000351	0.000374	0.001702	BJS
100	0.5	0.000729	0.060342	0.000634	0.000664	0.00069	0.000747	0.000784	0.000826	BJS
	1.5	0.000408	0.029735	0.000356	0.000375	0.000825	0.000421	0.000444	0.000951	BJS
	2.5	0.000294	0.02052	0.000258	0.000272	0.001097	0.000305	0.000322	0.001218	BJS
	3.5	0.000233	0.015866	0.000205	0.000216	0.001362	0.000243	0.000256	0.001478	BJS
	4.5	0.000195	0.013012	0.000171	0.000181	0.001602	0.000203	0.000214	0.001713	BJS
Best									Best	BJS

Table 4: MSE for all methods and different sample size for third case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.002314	0.053784	0.002112	0.002414	0.001468	0.002674	0.003269	0.00186	BJL
	1.5	0.00128	0.026481	0.001198	0.001426	0.000926	0.001575	0.001977	0.001551	BJL
	2.5	0.000921	0.018271	0.000872	0.001053	0.001058	0.00116	0.001471	0.001706	BJS
	3.5	0.00073	0.014125	0.000695	0.000847	0.001276	0.000931	0.001187	0.001914	BJS
	4.5	0.000608	0.011584	0.000581	0.000713	0.001498	0.000783	0.001002	0.002118	BJS
15	0.5	0.005126	0.074982	0.004973	0.005013	0.003792	0.004626	0.004807	0.003652	BGL
	1.5	0.002856	0.041052	0.002788	0.002861	0.001815	0.002653	0.002805	0.002056	BJL
	2.5	0.002061	0.029713	0.002019	0.002087	0.001521	0.001937	0.002061	0.001835	BJL
	3.5	0.001635	0.023666	0.001605	0.001666	0.001535	0.001546	0.001652	0.00187	BJL
	4.5	0.001364	0.019821	0.001341	0.001397	0.00164	0.001296	0.001389	0.001979	BGS
25	0.5	0.004499	0.063324	0.004385	0.004485	0.003452	0.004367	0.00452	0.003559	BJL
	1.5	0.002491	0.031031	0.002444	0.002528	0.001894	0.002467	0.002581	0.002123	BJL
	2.5	0.001794	0.021373	0.001765	0.001834	0.001697	0.00179	0.001881	0.001941	BJL
	3.5	0.001421	0.016508	0.001401	0.001459	0.001747	0.001424	0.0015	0.00199	BJS
	4.5	0.001185	0.013529	0.001169	0.00122	0.001866	0.001191	0.001256	0.002104	BJS
50	0.5	0.001517	0.057724	0.001489	0.001527	0.001291	0.001538	0.001593	0.001382	BJL
	1.5	0.000844	0.028366	0.000833	0.00086	0.000975	0.000867	0.000904	0.001101	BJS
	2.5	0.000609	0.019557	0.000602	0.000623	0.001123	0.000628	0.000656	0.001254	BJS

	3.5	0.000483	0.015113	0.000478	0.000495	0.001334	0.000499	0.000522	0.001464	BJS
	4.5	0.000402	0.01239	0.000399	0.000414	0.001546	0.000417	0.000437	0.001673	BJS
100	0.5	0.000193	0.067136	0.000198	0.000185	0.000121	0.000168	0.00016	0.00012	BJL
	1.5	0.000105	0.032902	0.000107	9.95E-05	0.000226	9.09E-05	8.68E-05	0.000267	BGP
	2.5	7.49E-05	0.022662	7.59E-05	7.08E-05	0.000502	6.49E-05	6.19E-05	0.000554	BGP
	3.5	5.91E-05	0.017503	5.98E-05	5.57E-05	0.000781	5.11E-05	4.88E-05	0.000836	BGP
	4.5	4.91E-05	0.014344	4.96E-05	4.63E-05	0.001038	4.25E-05	4.05E-05	0.001094	BGP
Best										BJS

Table 5: MSE for all methods and different sample size for fourth case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.002196	0.058551	0.002544	0.002182	0.002023	0.00232	0.002764	0.001597	BGL
	1.5	0.00119	0.029126	0.001321	0.001151	0.000491	0.001337	0.001642	0.001246	BJL
	2.5	0.00085	0.020183	0.000929	0.000816	0.000404	0.000978	0.001214	0.001406	BJL
	3.5	0.000671	0.015645	0.000727	0.000641	0.000547	0.000781	0.000977	0.001625	BJL
	4.5	0.000557	0.012853	0.0006	0.000531	0.000742	0.000655	0.000823	0.00184	BJP
15	0.5	0.005348	0.051505	0.004515	0.004605	0.003532	0.005127	0.005545	0.004244	BJL
	1.5	0.003112	0.025316	0.002586	0.002694	0.001805	0.003073	0.003377	0.002818	BJL
	2.5	0.002282	0.017457	0.001888	0.001983	0.001566	0.002278	0.00252	0.00258	BJL
	3.5	0.001826	0.013492	0.001508	0.001591	0.001602	0.001836	0.002037	0.002583	BJS
	4.5	0.001533	0.011062	0.001264	0.001338	0.001717	0.001548	0.001722	0.002661	BJS

25	0.5	0.00512	0.059228	0.004572	0.00467	0.003686	0.004997	0.005218	0.004265	BJL
	1.5	0.002937	0.029033	0.002604	0.002691	0.002055	0.002919	0.00308	0.002721	BJL
	2.5	0.002143	0.02	0.001896	0.001968	0.001813	0.002144	0.002272	0.002457	BJL
	3.5	0.001711	0.015448	0.001512	0.001574	0.001836	0.001718	0.001825	0.002451	BJS
	4.5	0.001433	0.012661	0.001266	0.00132	0.001939	0.001444	0.001536	0.002526	BJS
50	0.5	0.000854	0.060861	0.000781	0.000794	0.000652	0.000897	0.000948	0.000836	BJL
	1.5	0.000471	0.029935	0.000429	0.00044	0.000584	0.000503	0.000536	0.000839	BJS
	2.5	0.000339	0.020645	0.000308	0.000317	0.000812	0.000364	0.000389	0.001079	BJS
	3.5	0.000268	0.015957	0.000244	0.000251	0.001065	0.000289	0.000309	0.001331	BJS
	4.5	0.000223	0.013084	0.000203	0.000209	0.001304	0.000241	0.000258	0.001565	BJS
100	0.5	0.000344	0.064555	0.000354	0.000348	0.000277	0.000338	0.000342	0.000315	BJL
	1.5	0.000188	0.031699	0.000193	0.00019	0.000345	0.000186	0.000189	0.000445	BGS
	2.5	0.000135	0.021849	0.000138	0.000136	0.00061	0.000134	0.000136	0.000724	BGS
	3.5	0.000107	0.016881	0.000109	0.000107	0.000881	0.000106	0.000107	0.001	BGS
	4.5	8.86E-05	0.013839	9.04E-05	8.91E-05	0.001133	8.8E-05	8.95E-05	0.001251	BGP
Best										BJL

Table 6: MSE for all methods and different sample size for fifth case experiment.

n	T	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.014793	0.081456	0.013672	0.013504	0.01369	0.011425	0.01126	0.011548	BGP
	1.5	0.010726	0.059642	0.009953	0.01023	0.008332	0.008169	0.008356	0.006909	BGL
	2.5	0.008555	0.049046	0.007993	0.00836	0.006017	0.006507	0.006765	0.004926	BGL
	3.5	0.00719	0.042266	0.006754	0.007141	0.004743	0.00547	0.005744	0.003846	BGL
	4.5	0.00624	0.037422	0.005886	0.00627	0.003956	0.00475	0.005023	0.003188	BGL
15	0.5	0.006187	0.099123	0.006509	0.005889	0.007634	0.005879	0.00532	0.006911	BGP
	1.5	0.004039	0.056343	0.004152	0.003781	0.004009	0.003757	0.003418	0.003643	BGP
	2.5	0.003087	0.041029	0.00315	0.00288	0.002505	0.002852	0.002604	0.002272	BGL
	3.5	0.002529	0.032723	0.002572	0.002358	0.001707	0.002329	0.002131	0.001543	BGL
	4.5	0.002157	0.027408	0.002188	0.00201	0.001237	0.001982	0.001817	0.001115	BGL
25	0.5	0.002449	0.070873	0.002577	0.002365	0.003027	0.002396	0.002198	0.002817	BGP
	1.5	0.001677	0.040855	0.001722	0.001592	0.001694	0.001601	0.001478	0.001575	BGP
	2.5	0.001304	0.02988	0.00133	0.001234	0.001076	0.001236	0.001145	0.000997	BGL
	3.5	0.001079	0.023886	0.001096	0.00102	0.000746	0.001019	0.000946	0.000689	BGL
	4.5	0.000926	0.020035	0.000939	0.000875	0.000561	0.000872	0.000812	0.000519	BGL
50	0.5	0.003148	0.076593	0.003218	0.003097	0.003377	0.003111	0.002994	0.003265	BGP
	1.5	0.002094	0.044767	0.00212	0.002042	0.001903	0.00205	0.001975	0.00184	BGL
	2.5	0.001611	0.032899	0.001626	0.001568	0.001228	0.001573	0.001516	0.001187	BGL

	3.5	0.001325	0.026371	0.001336	0.001288	0.000865	0.001292	0.001246	0.000835	BGL
	4.5	0.001133	0.022159	0.001141	0.001101	0.000659	0.001104	0.001064	0.000636	BGL
100	0.5	0.001306	0.043052	0.001276	0.001292	0.001243	0.001249	0.001265	0.001217	BGL
	1.5	0.000939	0.026588	0.000923	0.00094	0.000868	0.000903	0.00092	0.000849	BGL
	2.5	0.000743	0.019977	0.000732	0.000748	0.000697	0.000716	0.000732	0.000683	BGL
	3.5	0.000621	0.016225	0.000613	0.000627	0.000625	0.0006	0.000613	0.000613	BGL
	4.5	0.000536	0.013758	0.00053	0.000543	0.000605	0.000519	0.000531	0.000595	BGS
Best										BGL

Table 7: MSE for all methods and different sample size for sixth case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.016982	0.092181	0.013807	0.01352	0.014382	0.011763	0.012096	0.01127	BGL
	1.5	0.014229	0.053877	0.010781	0.01104	0.009392	0.009709	0.010355	0.008193	BGL
	2.5	0.011985	0.03976	0.008914	0.009308	0.007015	0.00816	0.008835	0.006484	BGL
	3.5	0.010397	0.031982	0.007666	0.008102	0.005648	0.007075	0.007732	0.005453	BGL
	4.5	0.009219	0.026951	0.006763	0.007207	0.004774	0.006273	0.0069	0.004779	BJL
15	0.5	0.00875	0.06116	0.006192	0.006413	0.005956	0.006461	0.006933	0.005701	BGL
	1.5	0.006904	0.038101	0.004837	0.005189	0.004096	0.005233	0.005772	0.004285	BJL
	2.5	0.005659	0.028845	0.003966	0.004317	0.003155	0.004339	0.004841	0.00349	BJL
	3.5	0.004828	0.023557	0.003387	0.003719	0.002625	0.003728	0.004187	0.00303	BJL

	4.5	0.00423	0.02006	0.002971	0.003282	0.002304	0.003283	0.003704	0.002747	BJL
25	0.5	0.011591	0.089797	0.01003	0.010139	0.009852	0.009745	0.009953	0.009382	BGL
	1.5	0.010173	0.078087	0.008722	0.008951	0.008123	0.008526	0.00882	0.007901	BGL
	2.5	0.008701	0.073124	0.007452	0.007698	0.006753	0.007291	0.007586	0.006655	BGL
	3.5	0.007611	0.069604	0.006518	0.006761	0.005831	0.006379	0.006661	0.005809	BGL
	4.5	0.006785	0.066694	0.005812	0.006045	0.005186	0.005688	0.005954	0.005214	BJL
50	0.5	0.00173	0.044057	0.001681	0.001663	0.001713	0.001619	0.001632	0.001595	BGL
	1.5	0.00121	0.026261	0.001157	0.001155	0.001047	0.001137	0.00116	0.001025	BGL
	2.5	0.000948	0.019431	0.000903	0.000905	0.000745	0.000893	0.000916	0.00077	BJL
	3.5	0.000788	0.015634	0.000748	0.000752	0.000597	0.000744	0.000765	0.000652	BJL
	4.5	0.000678	0.013171	0.000643	0.000647	0.00053	0.000641	0.00066	0.000603	BJL
100	0.5	0.001008	0.067111	0.001088	0.001059	0.00112	0.001002	0.000982	0.001022	BGP
	1.5	0.000675	0.039489	0.00072	0.000701	0.00063	0.000668	0.000655	0.000588	BGL
	2.5	0.00052	0.029089	0.000553	0.000538	0.000407	0.000514	0.000505	0.000395	BGL
	3.5	0.000429	0.023348	0.000454	0.000442	0.000301	0.000423	0.000415	0.000308	BJL
	4.5	0.000367	0.019637	0.000388	0.000378	0.000259	0.000362	0.000355	0.000278	BJL
Best										BGL

Table 8: MSE for all methods and different sample size for seventh case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.014811	0.068945	0.013578	0.013434	0.013724	0.009386	0.009259	0.009629	BGP
	1.5	0.011199	0.046779	0.010315	0.010626	0.008804	0.006847	0.007002	0.005932	BGL
	2.5	0.009078	0.037052	0.008425	0.008833	0.006529	0.005491	0.005704	0.004278	BGL
	3.5	0.007703	0.031173	0.007191	0.007621	0.005239	0.004633	0.004859	0.003367	BGL
	4.5	0.006728	0.027128	0.00631	0.006738	0.004425	0.004033	0.004258	0.00281	BGL
15	0.5	0.002017	0.061925	0.00188	0.001891	0.002131	0.00147	0.001474	0.00168	BGS
	1.5	0.001424	0.035434	0.001334	0.001415	0.001222	0.001036	0.001094	0.00096	BGL
	2.5	0.001119	0.025854	0.001056	0.001145	0.000832	0.000818	0.000882	0.000651	BGL
	3.5	0.000932	0.020642	0.000884	0.000971	0.000646	0.000684	0.000746	0.000508	BGL
	4.5	0.000804	0.0173	0.000765	0.000848	0.000559	0.000591	0.000651	0.000448	BGL
25	0.5	0.000818	0.042399	0.000885	0.000788	0.001172	0.000763	0.000679	0.001011	BGP
	1.5	0.000567	0.025037	0.000591	0.000537	0.000628	0.000509	0.000461	0.000539	BGP
	2.5	0.000443	0.01847	0.000457	0.000419	0.000377	0.000393	0.00036	0.000321	BGL
	3.5	0.000368	0.014837	0.000377	0.000348	0.000259	0.000324	0.000298	0.000221	BGL
	4.5	0.000316	0.012486	0.000323	0.0003	0.000211	0.000278	0.000257	0.000184	BGL
50	0.5	0.001695	0.04662	0.001643	0.001659	0.001622	0.001507	0.001522	0.001489	BGL
	1.5	0.00124	0.027809	0.00121	0.001236	0.001113	0.001106	0.00113	0.001018	BGL
	2.5	0.000988	0.020583	0.000967	0.000993	0.000863	0.000883	0.000906	0.00079	BGL

	3.5	0.000829	0.016564	0.000814	0.000838	0.000741	0.000742	0.000764	0.000681	BGL
	4.5	0.000719	0.013956	0.000706	0.000729	0.000687	0.000644	0.000664	0.000636	BGL
100	0.5	0.001097	0.051817	0.001094	0.001086	0.0011	0.001052	0.001044	0.001058	BGP
	1.5	0.000765	0.030681	0.000762	0.00076	0.000698	0.000733	0.00073	0.00067	BGL
	2.5	0.000599	0.022648	0.000597	0.000596	0.000512	0.000573	0.000572	0.000492	BGL
	3.5	0.000497	0.018199	0.000496	0.000496	0.000428	0.000476	0.000476	0.000413	BGL
	4.5	0.000428	0.015318	0.000427	0.000427	0.0004	0.00041	0.00041	0.000388	BGL
Best										BGL

Table 9: MSE for all methods and different sample size for eighth case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.015043	0.120849	0.010756	0.010822	0.010972	0.008007	0.008548	0.007227	BGL
	1.5	0.012763	0.106073	0.008733	0.009216	0.007485	0.006649	0.007349	0.005393	BGL
	2.5	0.010782	0.099845	0.007312	0.007873	0.005722	0.005579	0.006255	0.00435	BGL
	3.5	0.009366	0.095971	0.006331	0.006899	0.004697	0.004829	0.00546	0.003733	BGL
	4.5	0.00831	0.093116	0.005609	0.006164	0.004042	0.004275	0.004862	0.003341	BGL
15	0.5	0.005035	0.062406	0.005646	0.005184	0.006514	0.003945	0.003853	0.00418	BGP
	1.5	0.003374	0.035767	0.003581	0.003333	0.003372	0.002637	0.002647	0.002324	BGL
	2.5	0.002605	0.026119	0.002713	0.002543	0.002109	0.002039	0.002071	0.001558	BGL
	3.5	0.002147	0.020864	0.002213	0.002084	0.001452	0.001683	0.001723	0.001169	BGL

	4.5	0.001838	0.017493	0.001882	0.001779	0.001074	0.001443	0.001485	0.000955	BGL
25	0.5	0.00418	0.062727	0.004067	0.003957	0.004294	0.003395	0.003407	0.003404	BGS
	1.5	0.003041	0.037676	0.002858	0.002829	0.002623	0.002468	0.002525	0.002217	BGL
	2.5	0.002419	0.02803	0.00225	0.002245	0.001841	0.001966	0.002028	0.001645	BGL
	3.5	0.002029	0.02264	0.001877	0.001882	0.001417	0.001651	0.001712	0.001339	BGL
	4.5	0.001757	0.019127	0.00162	0.00163	0.001171	0.001432	0.001489	0.001167	BGL
50	0.5	0.002837	0.074368	0.003191	0.003061	0.003366	0.002696	0.002606	0.002817	BGP
	1.5	0.001895	0.043016	0.002092	0.002007	0.001879	0.001787	0.001732	0.001606	BGL
	2.5	0.001461	0.031493	0.001602	0.001537	0.001201	0.001374	0.001333	0.001052	BGL
	3.5	0.001203	0.02519	0.001314	0.001262	0.000835	0.00113	0.001097	0.000758	BGL
	4.5	0.001029	0.021137	0.001121	0.001077	0.000627	0.000966	0.000938	0.000595	BGL
100	0.5	0.000188	0.034841	0.000173	0.000172	0.00018	0.000172	0.000179	0.000165	BGL
	1.5	0.000133	0.021337	0.000121	0.000122	0.000112	0.000123	0.00013	0.000117	BJL
	2.5	0.000104	0.015938	9.5E-05	9.59E-05	9.46E-05	9.73E-05	0.000103	0.000115	BJL
	3.5	8.7E-05	0.012893	7.9E-05	8E-05	0.000111	8.13E-05	8.66E-05	0.000143	BJS
	4.5	7.49E-05	0.0109	6.81E-05	6.9E-05	0.000149	7.02E-05	7.5E-05	0.000188	BJS
Best										BGL

Table 10: MSE for all methods and different sample size for ninth case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.015538	0.101191	0.014103	0.013792	0.015118	0.011586	0.01135	0.012414	BGP
	1.5	0.015229	0.143665	0.013676	0.013892	0.013071	0.010805	0.010991	0.010331	BGL
	2.5	0.013859	0.157969	0.012506	0.012925	0.011199	0.009689	0.010025	0.008654	BGL
	3.5	0.012637	0.163056	0.011464	0.011978	0.009808	0.008769	0.009172	0.007459	BGL
	4.5	0.011615	0.164091	0.010587	0.011149	0.008753	0.008023	0.008457	0.006576	BGL
15	0.5	0.010132	0.093303	0.009038	0.009136	0.009079	0.007778	0.007877	0.007795	BGS
	1.5	0.009766	0.114051	0.008842	0.009199	0.008211	0.007477	0.00779	0.006935	BGL
	2.5	0.008733	0.122333	0.007994	0.008423	0.0071	0.006708	0.007077	0.005951	BGL
	3.5	0.00786	0.126318	0.00725	0.007701	0.006245	0.006055	0.006439	0.005208	BGL
	4.5	0.007153	0.128272	0.006636	0.007088	0.005593	0.005524	0.005906	0.004649	BGL
25	0.5	0.00809	0.086197	0.008347	0.007909	0.009232	0.007818	0.007408	0.00865	BGP
	1.5	0.006154	0.087145	0.006246	0.005945	0.006418	0.005857	0.005572	0.00603	BGP
	2.5	0.005021	0.084104	0.005072	0.004841	0.004795	0.004757	0.004538	0.004508	BGL
	3.5	0.004283	0.080642	0.004316	0.004127	0.003754	0.004048	0.003868	0.003528	BGL
	4.5	0.003757	0.077301	0.00378	0.00362	0.003033	0.003546	0.003393	0.002849	BGL
50	0.5	0.003378	0.038941	0.003281	0.003285	0.003289	0.003153	0.003156	0.00316	BGS
	1.5	0.002793	0.032968	0.002723	0.002754	0.002609	0.002613	0.002644	0.002506	BGL
	2.5	0.002352	0.028492	0.002299	0.002336	0.002118	0.002206	0.002241	0.002033	BGL

	3.5	0.002043	0.025245	0.002	0.002039	0.001785	0.001919	0.001956	0.001714	BGL
	4.5	0.001814	0.022769	0.001779	0.001818	0.001553	0.001706	0.001743	0.00149	BGL
100	0.5	0.00253	0.035416	0.002449	0.002482	0.002384	0.002399	0.002431	0.002336	BGL
	1.5	0.002106	0.032287	0.002054	0.002096	0.001954	0.002011	0.002052	0.001913	BGL
	2.5	0.001776	0.028537	0.001739	0.00178	0.001634	0.001703	0.001742	0.0016	BGL
	3.5	0.001545	0.025571	0.001516	0.001554	0.001422	0.001484	0.001521	0.001393	BGL
	4.5	0.001373	0.023221	0.001349	0.001385	0.001279	0.001321	0.001355	0.001254	BGL
Best										BGL

Table 11: MSE for all methods and different sample size for tenth case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.022718	0.100064	0.018233	0.017873	0.019251	0.015808	0.016056	0.015559	BGL
	1.5	0.023208	0.109145	0.017175	0.017554	0.015955	0.015724	0.016518	0.014049	BGL
	2.5	0.021411	0.111098	0.01551	0.016157	0.013415	0.014401	0.015355	0.012242	BGL
	3.5	0.019672	0.110637	0.014111	0.014877	0.011617	0.013184	0.01419	0.010851	BGL
	4.5	0.018173	0.109139	0.012963	0.013785	0.010288	0.012153	0.013168	0.009784	BGL
15	0.5	0.006402	0.091479	0.00851	0.007645	0.010463	0.005878	0.005461	0.006923	BGP
	1.5	0.004972	0.076954	0.00616	0.005554	0.006934	0.004457	0.004212	0.004784	BGP
	2.5	0.004092	0.068198	0.004942	0.00447	0.005038	0.003645	0.003477	0.003562	BGP
	3.5	0.003508	0.061859	0.004177	0.003787	0.003854	0.003116	0.002991	0.002785	BGL

	4.5	0.003087	0.05692	0.003641	0.003308	0.003048	0.002739	0.002641	0.002252	BGL
25	0.5	0.006294	0.047391	0.005751	0.005679	0.005965	0.005423	0.005468	0.005395	BGL
	1.5	0.005451	0.045617	0.004811	0.004843	0.004639	0.004714	0.004848	0.0044	BGL
	2.5	0.00468	0.041551	0.00409	0.004155	0.003724	0.004063	0.004216	0.003633	BGL
	3.5	0.004114	0.037977	0.003578	0.003655	0.003107	0.003582	0.003737	0.003104	BGL
	4.5	0.003683	0.034988	0.003195	0.003276	0.00267	0.003215	0.003367	0.002726	BGL
50	0.5	0.004465	0.061513	0.004516	0.004444	0.004657	0.004244	0.004206	0.004322	BGP
	1.5	0.003584	0.059581	0.003551	0.003518	0.003469	0.0034	0.003397	0.003293	BGL
	2.5	0.002984	0.05449	0.002937	0.00292	0.002705	0.002833	0.002841	0.002607	BGL
	3.5	0.002575	0.049936	0.002526	0.002516	0.002199	0.002446	0.002459	0.002151	BGL
	4.5	0.002277	0.046094	0.002228	0.002223	0.001846	0.002164	0.002179	0.001831	BGL
100	0.5	0.000905	0.021654	0.000927	0.000913	0.00096	0.000882	0.000876	0.000898	BGP
	1.5	0.000751	0.017466	0.000758	0.000749	0.000755	0.00073	0.000729	0.000718	BGL
	2.5	0.000633	0.014651	0.000636	0.000629	0.000599	0.000615	0.000616	0.000581	BGL
	3.5	0.00055	0.012717	0.000551	0.000546	0.000493	0.000535	0.000536	0.000488	BGL
	4.5	0.000488	0.011294	0.000488	0.000485	0.00042	0.000475	0.000477	0.000425	BJL
Best										BGL

Table 12: MSE for all methods and different sample size for eleventh case experiment.

n	t	MLE	MOM	BJS	BJP	BJL	BGS	BGP	BGL	Best
10	0.5	0.016562	0.040899	0.013535	0.013722	0.01347	0.011136	0.011325	0.011038	BGL
	1.5	0.015357	0.049867	0.013275	0.014581	0.01057	0.010443	0.011495	0.008319	BGL
	2.5	0.010414	0.037928	0.009362	0.010619	0.006854	0.007229	0.00821	0.005314	BGL
	3.5	0.007243	0.028066	0.006663	0.007692	0.005025	0.005093	0.005883	0.003926	BGL
	4.5	0.005268	0.021223	0.004919	0.005743	0.004204	0.003737	0.004363	0.003372	BGL
15	0.5	0.008761	0.034408	0.008457	0.008132	0.009337	0.007551	0.007264	0.008335	BGP
	1.5	0.006036	0.02204	0.005732	0.00583	0.005145	0.005072	0.005146	0.004586	BGL
	2.5	0.003731	0.01365	0.003579	0.003736	0.002622	0.003152	0.003277	0.002319	BGL
	3.5	0.002478	0.009114	0.002396	0.002537	0.001658	0.002104	0.002218	0.00147	BGL
	4.5	0.001754	0.006478	0.001705	0.001822	0.001408	0.001495	0.00159	0.001277	BGL
25	0.5	0.005443	0.020308	0.005675	0.005363	0.006418	0.005266	0.004978	0.005953	BGP
	1.5	0.004018	0.014436	0.004003	0.00387	0.003956	0.003688	0.003562	0.003651	BGP
	2.5	0.002535	0.009223	0.002516	0.002463	0.001954	0.00231	0.002258	0.001787	BGL
	3.5	0.001701	0.006256	0.001689	0.001665	0.001121	0.001547	0.001523	0.001024	BGL
	4.5	0.001211	0.004489	0.001203	0.001192	0.000915	0.001101	0.001088	0.000853	BGL
50	0.5	0.002485	0.006208	0.002572	0.002484	0.002774	0.002482	0.002397	0.002676	BGP
	1.5	0.001598	0.004434	0.001612	0.001565	0.001544	0.001556	0.001511	0.001491	BGL
	2.5	0.00096	0.002762	0.000964	0.00094	0.000694	0.000931	0.000907	0.00067	BGL

	3.5	0.000628	0.001841	0.00063	0.000615	0.00044	0.000608	0.000594	0.000429	BGL
	4.5	0.00044	0.001305	0.000441	0.000432	0.000525	0.000426	0.000417	0.00052	BGP
100	0.5	0.001249	0.005819	0.001162	0.001193	0.001105	0.001138	0.001169	0.001082	BGL
	1.5	0.000962	0.004763	0.000928	0.000969	0.00087	0.000908	0.000948	0.000852	BGL
	2.5	0.000609	0.003082	0.000595	0.000625	0.000682	0.000581	0.000611	0.00067	BGS
	3.5	0.000409	0.002092	0.000402	0.000423	0.000758	0.000392	0.000414	0.000748	BGS
	4.5	0.000291	0.001499	0.000287	0.000303	0.000991	0.00028	0.000296	0.000983	BGS
Best										BGL

Discussion

1. From the above tables we can conclude that Bayes method depending on Jeffery information and squared loss function (BJS) was the best estimator of survival function for small (ρ) values and Bayes method of Gamma function and loss function (Linex), also (BGL) was the best estimator of survival function for large (ρ) values for all sample sizes.
2. MSE values were decreasing with increasing of sample size for all methods and cases that agreed with statistical theory, table (13) below summarise the above tables.

Table 13: Best MSE methods of simulation procedure

Table	2	3	4	5	6	7	8	9	10	11
MSE	BJS	BJS	BJS	BJL	BGL	BGL	BGL	BGL	BGL	BGL

Conclusion

We can depend on Bayes method by Jeffery information and quadratic loss function (BJS) in estimation of survival function for small (ρ) values and Bayes method of Gamma function information and [Linex (BGL)] in survival function estimator of large (ρ) values for all sample sizes. Also, we can improve this paper in case of missing data and data under control or under sequence control.

REFERENCES

ALAA A, (2015), "ESTIMATION OF R = P[Y < X} FOR THE THREE PARAMETER DAGUM DISTRIBUTION", Journal of Statistics: Advances in Theory and Applications 13,No (2), , pp 167-178.

Bander, Al-Z and Samerah, B, (2016), "Estimation of the Stress-Strength Reliability for the Dagum Distribution", Journal of Advanced Statistics, 1, No (6); pp 152 - 166

BRODERICK, O.O, SAS Lee, E.T. (2017)ITH, R, (2013), "THE MC-DAGUM DISTRIBUTION AND ITS STATISTICAL PROPERTIES WITH APPLICATIONS", ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS. 4, No (84); pp 173 - 186

Domma, F., Giordano, S. and Zenga, M. M. (2011). Maximum likelihood estimation in Dagum distribution with censored sample, Journal of Applied Statistics, 38, No (12), pp. 2971-2985.

Lee, E.T. (2017) *Statistical Methods for Survival Data Analysis* .Lifetime, learning publications .Inc. Belmount , 6,No (51) ; pp 162 - 173

Oluyede, B.O. & Rajasooriya, S. (2013). The Mc-Dagum distribution and its statistical proper ties with applications, Asian Journal of Mathematics and Applications, 8,No (44); p. 1-16.

Omari,M.A, Kutubi,H,S and Ibrahim,N,A (2010),"Comparison of the Bayesian and Maximum Likelihood Estimation for Weibull Distribution"J.Math&stat. 6, No (2); pp 100-104.

Saima N, S. P. Ahmad and Aquil A, (2017), "BAYESIAN ANALYSIS OF DAGUM DISTRIBUTION",Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666 : 10,No (1); pp123-136.

Sanku D, Bander Al-Z & Samerah B, (2017), "Dagum Distribution: Properties and Different Methods of Estimation", International Journal of Statistics and Probability; 6, No (2) ; pp 168 - 188.

Shahzad, N, M. and Asghar, Z. (2015). "Transmuted Dagum distribution: A more exible and broad shaped hazard function model" Mathematics and Statistics February. 4, No (52) ; pp 237 - 253

Shahzad, M. N. and Asghar, Z. (2013). Comparing TL-moments, L-moments and conventional moments of Dagum distribution by simulated data, Revista Colombiana de Estadistica, 1, No (36), pp. 79-93.